Systems Analysis and Control

Matthew M. Peet
Arizona State University

Lecture 3: Linearization
Introduction

In this Lecture, you will learn:

How to Linearize a Nonlinear System. System.

• Taylor Series Expansion
• Derivatives
• L’hopital’s rule
• Multiple Inputs/ Multiple States
Let's **Start** with an Example

A Simple Pendulum

Consider the rotational dynamics of a pendulum:

- The **input** is a motor-driven moment, \( T(t) \).
- The **output** is the angle, \( \theta(t) \).
- The moment of inertia about the pivot point is \( J \).
- The force of gravity, \( Mg \) acts on the center of mass.
  - Force creates a moment about the pivot (See Figure b)):

\[
N(t) = -Mg \sin \theta(t) \cdot \frac{l}{2}
\]
A Simple Pendulum

The governing equation is Newton’s law:

\[ \ddot{\theta}(t) = \frac{N(t)}{J} + T(t) \]

Equations of Motion (EOM):

\[ \ddot{\theta}(t) = -\frac{Mgl}{2J} \sin \theta(t) + \frac{T(t)}{J} \]

\[ y(t) = \theta(t) \]

First-order form: Let \( x_1(t) = \theta(t) \), \( x_2(t) = \dot{\theta}(t) \).

\[ \dot{x}_1(t) = x_2(t) \]

\[ \dot{x}_2(t) = -\frac{Mgl}{2J} \sin x_1(t) + \frac{T(t)}{J} \]

\[ y(t) = x_1(t) \]
A Simple Pendulum

The Problem

First-order form:

\[ \dot{x}_1(t) = x_2(t) \]
\[ \dot{x}_2(t) = -\frac{Mgl}{2J} \sin x_1(t) + \frac{T(t)}{J} \]
\[ y(t) = x_1(t) \]

Although we have the system in first-order form, it cannot be put in state-space because of the \( \sin x_1 \) term.

What to do???

Although \( \sin x \) is nonlinear, small sections look linear.

- **Near** \( x = 0 \): \( \sin x \cong x \)
- **Near** \( x = \pi/2 \): \( \sin x \cong 1 \)
- **Near** \( x = \pi \): \( \sin x \cong \pi - x \)

We must use these linear approximations very carefully!
Accuracy of the Small Angle Approximation

The approximation will only be accurate for a narrow band of $x$.

- **80% Accuracy**: $x \in [-1.2, 1.2]$
- **95% Accuracy**: $x \in [-.7, .7]$

- **80% Accuracy**: $x \in [.9, 2.2]$
- **95% Accuracy**: $x \in [1.25, 1.9]$
Linear Approximation

We can use the tangent to approximate a nonlinear function near a point $x_0$.

**Key Point:** The approximation is \textit{tangent} to the function at the point $x_0$.

\[
f(x) \approx ax + b
\]

- The slope is given by
  \[
a = \left. \frac{d}{dx} f(x) \right|_{x=x_0}
\]
- The \textit{y-intercept} is given by
  \[
b = f(x_0) - ax_0
\]

The \textbf{linear approximation} is given by

\[
f(x) \approx f(x_0) + \left. \frac{d}{dx} f(x) \right|_{x=x_0} (x - x_0)
\]
Problem: Approximate the scalar function $f(x)$ near the point $x_0$ using

$$y(x) = ax + b$$

The Linear Approximation is given by

$$y(x) = f(x_0) + \frac{d}{dx}f(x)\big|_{x=x_0}(x - x_0)$$
Linear Approximation

**Note:** The Linear Approximation is just the first two terms in the Taylor Series representation.

\[ f(x) = f(x_0) + \frac{d}{dx} f(x)\bigg|_{x=x_0} \frac{(x - x_0)}{1!} + \frac{d}{dx} f(x)\bigg|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots \]
Example: Pendulum

Return to the dynamics of a pendulum:

\begin{align*}
\dot{x}_1(t) & = x_2(t) \\
\dot{x}_2(t) & = -\frac{Mgl}{2J} \sin x_1(t) + \frac{1}{J} T(t) \\
y(t) & = x_1(t)
\end{align*}

The nonlinear term is \( \sin x_1 \)

- We want to linearize \( \sin x_1 \).
- **Choose an operating point, \( x_0 \)!**
  - Depends on what we want to do!
  - Options are limited.

**Hanging Pendulum:** \( x_0 = 0 \)

**Inverted Pendulum:** \( x_0 = \pi \)

**Tracking:** \( x_0 = ??? \)
Example: Balance an Inverted Pendulum

For the inverted pendulum: \( f(x) = \sin x \) and \( x_0 = \pi \).

- **Slope:**
  \[
  a = \left. \frac{d}{dx} f(x) \right|_{x=x_0} = \cos(\pi) = -1
  \]

- \( f(x_0) = f(\pi) = \sin(\pi) = 0 \)
- Thus (for \( x \approx \pi \))
  \[
  \sin(x) \approx f(x_0) + a(x - x_0) = 0 - 1(x - \pi) = \pi - x
  \]

This gives the first-order dynamics:
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= \frac{Mgl}{2J} x_1(t) - \frac{Mgl}{2J} \pi + \frac{1}{J} T(t) \\
y(t) &= x_1(t)
\end{align*}
\]

**New Problem:** How to eliminate the constant term \(-\frac{Mgl}{2J} \pi\)
Solution: Measure Displacement from Equilibrium

Define the state variable as Displacement from Equilibrium

Definition 1.

$x_0$ is an **Equilibrium Point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. (Then $\dot{x} = f(x_0) = 0$)

- Nonlinear systems may have *many* equilibrium points.
- Linear systems only have one equilibrium point ($x_0 = 0$).
A Change of Variables

Consider distance from equilibrium $\Delta x = x - x_0$

The pendulum has infinite equilibria.

- **Down equilibria**: $\theta_0 = 0 + 2\pi n$ for $n = 1, \cdots, \infty$
- **Up equilibria**: $\theta_0 = \pi + 2\pi n$ for $n = 1, \cdots, \infty$

**Lets choose** $\theta_0 = \pi$ i.e. $x_0 = [\pi \ 0]^T$:

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = \frac{Mgl}{2J}(x_1(t) - \pi)$$

**Problem:** Translate the equilibrium to $\Delta x_0 = 0$.

**Solution:** Define a new variable $\Delta x = x - x_0$
($\Delta x_1 = x_1 - \pi$, $\Delta x_2 = x_2$)

- Then
  $$\Delta \dot{x}_1(t) = \dot{x}_1(t) = x_2(t) = \Delta x_2(t)$$
  $$\Delta \dot{x}_2(t) = \dot{x}_2(t) = \frac{Mgl}{2J}(x_1(t) - \pi) = \frac{Mgl}{2J} \Delta x_1(t)$$

- Thus $\Delta x_0 = 0$ is the equilibrium!!!
Measuring Displacement from Equilibrium

Pendulum Example

Summary:

• Linearize about equilibrium $x_1 = \pi$, $x_2 = 0$. This yields

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{Mgl}{2J} (x_1(t) - \pi) + \frac{1}{J} T(t)$$

• Define new states as displacement from equilibrium:

$$\Delta x_1(t) = x_1(t) - \pi$$

$$\Delta x_2(t) = x_2(t)$$

• Get the new dynamics:

$$\Delta \dot{x}_1(t) = \Delta x_2(t)$$

$$\Delta \dot{x}_2(t) = \frac{Mgl}{2J} \Delta x_1(t) + \frac{1}{J} T(t)$$

$\Delta x_1$ is angle from the vertical.
Now we are ready for state-space.

**New Dynamics:**

\[
\Delta \dot{x}_1(t) = \Delta x_2(t)
\]
\[
\Delta \dot{x}_2(t) = \frac{Mgl}{2J} \Delta x_1(t) + \frac{1}{J} T(t)
\]

**State-Space Form:**

\[
\Delta \dot{x}(t) = \begin{bmatrix}
0 & 1 \\
\frac{Mgl}{2J} & 0
\end{bmatrix} \Delta x(t) + \begin{bmatrix}
0 \\
\frac{1}{J}
\end{bmatrix} T(t)
\]
\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta x(t)
\]

\[A = \begin{bmatrix}
0 & 1 \\
\frac{Mgl}{2J} & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\frac{1}{J}
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}\]
Example: Balance an Inverted Pendulum

**Applications:** Walking robots.
Example: Balance an Inverted Pendulum

**Applications:** Segway.
Numerical Example: Using l’Hôpital’s rule

Occasionally you will encounter a system such as

\[ \ddot{x}(t) = -\dot{x}(t) + \frac{\sin^2(x(t))}{x(t)} \]

where you want to linearize about the zero equilibrium.

The nonlinear term is \( \frac{\sin^2 x}{x} \) with equilibrium point \( x_0 = 0 \).

Recall the formula:

\[ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \]

Thus we must calculate \( f(x_0) \) and \( f'(x_0) \).

Lets start with \( f(x_0) \). Initially, we see that \( f(0) = \frac{0}{0} \), which is indeterminate. To help, we use L’hôpital’s Rule.
**Theorem 2 (l’Hôpital’s Rule).**

If \( g(0) = 0 \) and \( h(0) = 0 \), then

\[
\lim_{x \to 0} \frac{g(x)}{h(x)} = \lim_{x \to 0} \frac{g'(x)}{h'(x)}
\]

Let's linearize \( f(x) = \frac{\sin^2 x}{x} \) about \( x_0 = 0 \). We need to find \( f(x_0) \) and \( f'(x_0) \).

\[
f(0) = \lim_{x \to 0} \left( \frac{g(x)}{h(x)} \right) = \lim_{x \to 0} \frac{\sin^2(x)}{x} \quad \frac{g'(x)}{h'(x)} = \frac{2 \sin x \cos x}{1} = \frac{0}{1} = 0
\]

which is as expected. Now,

\[
f'(x) = \frac{2 \sin x \cos x}{x} - \frac{\sin^2 x}{x^2} = \frac{g'(x)}{h'(x)} = \frac{2x \sin x \cos x - \sin^2 x}{x^2}
\]

So unfortunately,

\[
f'(0) = \frac{0}{0}
\]
Example Continued

So once more we apply L’hopital’s rule:

\[
\frac{2 \sin x \cos x + 2x \cos^2 x - 2x \sin^2 x - 2 \sin x \cos x}{f'(x)} = \frac{2x}{g'(x)}
\]

\[
= \lim_{x \to 0} \frac{2x \left( \cos^2 x - \sin^2 x \right)}{h'(x)} = 0
\]

Ooops, we must apply l’Hôpital’s rule AGAIN:

\[
\lim_{x \to 0} \frac{2 \cos^2 x - \sin^2 x}{2x} = \lim_{x \to 0} \frac{2 \left( \cos^2 x - \sin^2 x \right) - 8x \cos x \sin x}{2} = \frac{2}{2} = 1
\]

Which was a lot of work for such a simple answer (easier way?). We have the linearized equation of motion:

\[
\ddot{x}(t) = -\dot{x}(t) + 0 + 1 \cdot (x(t) - 0)
\]

For state-space, let \(x_1 = x\), \(x_2 = \dot{x}\). Then \(\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t)\).
Inverted Pendula Videos

Double Inverted Pendulum

Triple Inverted Pendulum

This is actually a switched controller!

1. Linearize about hanging position
   1.1 Design Energy Maximizing Controller
   1.2 Will cause pendulum to swing up.

2. Linearize about inverted position
   2.1 Switch controllers when near inverted position
What have we learned today?

How to Linearize a Nonlinear System System.

- Taylor Series Expansion
- Derivatives
- L’hoptial’s rule
- Multiple Inputs/ Multiple States

Next Lecture: Laplace Transform