Systems Analysis and Control

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Lecture 4: The Laplace Transform (and Friends)

Introduction

In this Lecture, you will learn:

Simple ways to use state-space.

- How to find the output given an input.
- Linearity

Linear Systems.

- The Fourier Series
  - Representing signals as the sum of sinusoids.
  - Representing systems using response to sinusoids.
- The Fourier Transform
- The Laplace Transform
  - Representing signals in the frequency domain.
  - Representing systems using response to sinusoids.

How to represent a system using a *Transfer Function*.

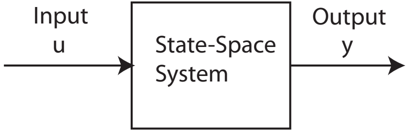
- How to find the output given an input.

Recall the state-space form

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



Basic Question: Given an input function, $u(t)$, what is the output?
Solution: Solve the differential Equation.

Example: The equation

$$\dot{x}(t) = ax(t), \qquad x(0) = x_0$$

has solution

$$x(t) = e^{at}x_0,$$

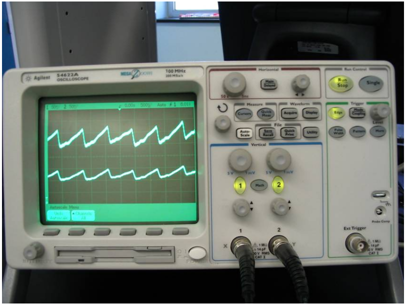
But we are interested in Matrices!!!

Not a rule, but sometimes... If it works for scalars, it also works for matrices.

The Solution to State-Space

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



The equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

Solution:

$$x(t) = e^{At}x_0$$

The Matrix exponential is defined by the series expansion

$$e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots$$

Don't Worry! You will never have to calculate a matrix exponential by hand. The important part is that

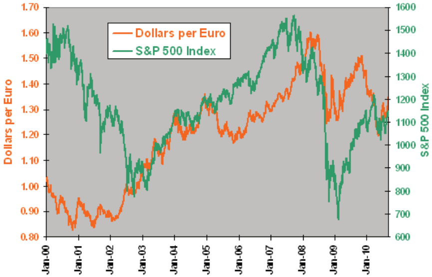
$$\dot{x}(t) = \frac{d}{dt}e^{At}x_0 = Ae^{At}x_0 = Ax(t), \qquad \text{and} \qquad x(0) = e^{0}x_0 = x_0$$

What happens when we add an input instead of an initial condition?

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



The equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0$$

Solution:

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

Proof.

Check the solution:

$$x(t) = \int_0^t e^{A(t-s)}Bu(s)ds$$
$$\dot{x}(t) = e^0 Bu(t) + A\int_0^t e^{A(t-s)}Bu(s)ds = Bu(t) + Ax(t)$$

Calculating the Output

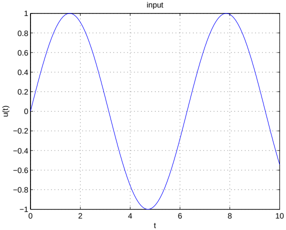
Numerical Example, $u(t) = \sin(t)$

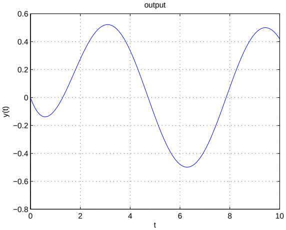
State-Space:

$$\dot{x} = -x(t) + u(t)$$
$$y(t) = x(t) - .5u(t) \qquad x(0) = 0$$
$$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$$

Solution:

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \\ &= e^{-t}\int_0^t e^s \sin(s)ds - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^s(\sin s - \cos s)|_0^t\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^t(\sin t - \cos t) + 1\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}\left(e^{-t} - \cos t\right) \end{aligned}$$





Problem: Finding the integral of complicated functions is tough.
Solution: Use the linearity of the system.

A system G is a MAP from input functions to output functions. e.g., $G : u \mapsto y$

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

Definition 1.

A **SYSTEM is Linear** if for any scalars, α and β, and any inputs u_1 and u_2,

$$G(\alpha u_1 + \beta u_2) = \alpha G u_1 + \beta G u_2$$

State-space systems are LINEAR.

Properties of Linearity

Properties: Multiplication

- $y = Gu$ implies $\alpha y = G(\alpha u)$.
- larger input implies a larger output.

Properties: Addition

- $y_1 = Gu_1$ and $y_2 = Gu_2$ implies $G(u_1 + u_2) = y_1 + y_2$.
- Can calculate complicated outputs by addition.
  - Step 1: Decompose input into simple parts: $u(t) = u_1(t) + u_2(t)$
  - Step 2: Calculate simple outputs: y_1 and y_2.
  - Step 3: Calculate the complicated output: $y(t) = y_1(t) + y_2(t)$

Example: Using previous system,

$$\begin{aligned}\dot{x} &= -x(t) + u(t) \\ y(t) &= x(t) - .5u(t) \qquad x(0) = 0\end{aligned}$$

Let $u(t) = 2\sin(t) - \cos(t)$.

Question: Can we compute the output without recalculating the integral???

Example of Linearity

Recall our simple state-space system:

$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$

- We found the that input $u_1(t) = \sin(t)$ produces output $y_1(t) = \frac{1}{2}\left(e^{-t} - \cos t\right)$.

Now if $u_2(t) = \cos(nt)$, we have

$$\begin{aligned}
y_2(t) &= \int_0^t Ce^{A(t-s)}Bu_2(s)ds + Du_2(t) \\
&= e^{-t}\int_0^t e^s \cos(ns)ds - \frac{1}{2}\cos(nt) \\
&= \frac{1}{2}e^{-t}\int_0^t \left(e^{(1+\imath n)s} + e^{(1-\imath n)s}\right)ds - \frac{1}{2}\cos(nt) \\
&= \frac{1}{2}e^{-t}\left[\frac{1}{(1+\imath n)}e^{(1+\imath n)s} + \frac{1}{(1-\imath n)}e^{(1-\imath n)s}\right]_0^t - \frac{1}{2}\cos(nt) \\
&= \frac{-e^{-t} + \cos(nt) + n\sin(nt)}{1+n^2} - \frac{1}{2}\cos(nt)
\end{aligned}$$

Example of Linearity

For our simple state-space system:

- We found the that input $u_1(t) = \sin(t)$ produces output $y_1(t) = \frac{1}{2}\left(e^{-t} - \cos t\right)$.

Now if $u_2(t) = \cos(t)$, we have

$$y_2(t) = \frac{1}{2}\left(\sin t - e^{-t}\right)$$

So for the composite input, $u(t) = 2u_1(t) - u_2(t)$, we find

$$\begin{aligned} y(t) &= 2y_1(t) - y_2(t) \\ &= \left(e^{-t} - \cos t\right) - \frac{1}{2}\left(\sin t - e^{-t}\right) \\ &= \frac{3}{2}e^{-t} - \cos t - \frac{1}{2}\sin t \end{aligned}$$

Frequency Domain version 1: Fourier Series

We calculated the response to a complicated signal

$$u(t) = 2\sin(t) - \cos(t)$$

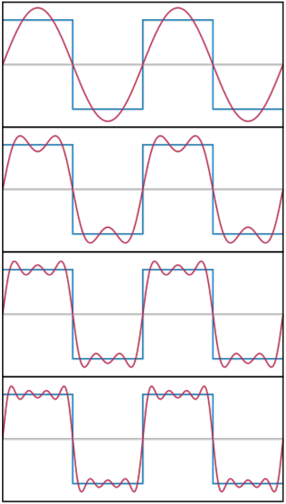
by adding the responses to the simple sinusoids:

$$u_1(t) = \sin(t) \qquad \text{and} \qquad u_2(t) = \cos(t)$$

So that $y(t) = y_1(t) + y_2(t)$

Question: What about more complicated signals???

Claim: We can recreate ANY signal $u(t)$ by adding up sinusoids.

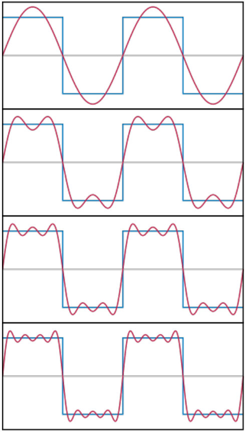


Conclusion: Since we know the output when the input is sinusoid, we can calculate the solution for ANY input signal.

Frequency Domain version 1: Fourier Series

Question: How do we express our input using sines and cosines???

Answer: The Fourier Series.



Theorem 2 (Fourier Series).

Any input signal, u on the time interval $[-\pi, \pi]$ is the combination of sines and cosines:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \sin(nt) + b_n \cos(nt)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(s)\sin(ns)ds \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(s)\cos(ns)ds$$

Frequency Domain version 1: The Fourier Series

Assume we know the output produced by sinusoids:

- $y_{1,n}(t)$ is the output from $u_{1,n}(t) = \sin(nt)$
- $y_{2,n}(t)$ is the output from $u_{2,n}(t) = \cos(nt)$

Then for the input

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \sin(nt) + b_n \cos(nt)]$$

The output is

$$y(t) = \frac{b_0}{2} y_{2,0}(t) + \sum_{n=1}^{\infty} [a_n y_{1,n}(t) + b_n y_{2,n}(t)]$$

Conclusion: The functions $y_n(t)$ make it very easy to calculate the output for any input.
Problem: We need an infinite number of a_n and $y_n(t)$.

Frequency Domain version 2: The Fourier Transform

An alternative to the Fourier Series is the Fourier Transform.

Instead of a **Sum** of sinusoids, we express the signal as an **Integral** of sinusoids:

$$u(t) = \int_0^\infty a(\omega)\sin(\omega t)d\omega + \int_0^\infty b(\omega)\cos(\omega t)d\omega$$

Alternatively, (using $\sin(\omega t) = \frac{e^{\imath\omega t} - e^{-\imath\omega t}}{2i}$ and $\cos(\omega t) = \frac{e^{\imath\omega t} + e^{-\imath\omega t}}{2}$):

$$u(t) = \int_{-\infty}^\infty \hat{u}(\imath\omega)e^{\imath\omega t}d\omega$$

Here $\hat{u}(\imath\omega)$ is the **Fourier Transform** of the signal $u(t)$ and is computed as follows:

Theorem 3.

The **Fourier Transform** *of the signal* u *can be found as*

$$\hat{u}(\imath\omega) = \int_{-\infty}^\infty e^{-\imath\omega t}u(t)dt$$

Inverse Fourier Transform

Recall: $\hat{u}$ is the Fourier Transform of u if

$$\hat{u}(\imath\omega) = \int_{-\infty}^{\infty} e^{-\imath\omega t} u(t) dt$$

In order to recover u from $\hat{u}$, we can use the following:

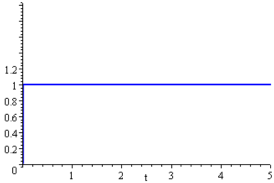
Theorem 4 (Inverse Fourier Transform).

Given $\hat{u}$, the Fourier Transform of u, we can find u as

$$u(t) = \int_{-\infty}^{\infty} e^{\imath\omega t} \hat{u}(\imath\omega) d\omega$$

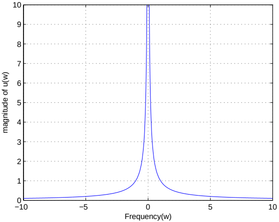
Consider the input signal:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$$



The Fourier transform is

$$\begin{aligned}\hat{u}(\imath\omega) &= \int_0^\infty e^{-\imath\omega t} dt \\ &= \frac{1}{-\imath\omega}\left[e^{-\imath\omega t}\right]_0^\infty = \frac{1}{-\imath\omega}\end{aligned}$$

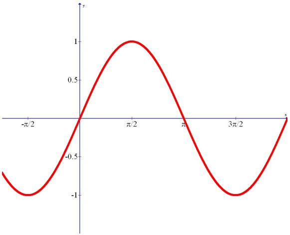


The Fourier Transform

Example

Consider the input signal:

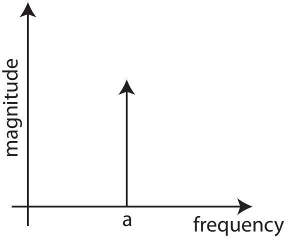
$$u(t) = e^{\imath a t}$$



The Fourier transform is

$$\hat{u}(\imath\omega) = \begin{cases} \infty & \omega = a \\ 0 & \omega \neq a \end{cases}$$
$$= \delta(\omega - a)$$

δ is the Dirac Delta function.



We'll do many more examples next lecture.

The Fourier Transform

The Fourier transform can be applied to samples of music to understand the frequency composition.

Application: The Graphic Equalizer:

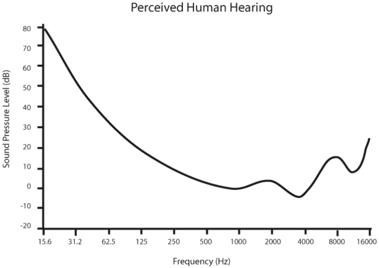


Figure: Equal Loudness Test

Audio Compression

- Sound is communicated in compression waves
- Humans can only hear sound in the range 20Hz-20kHz
- Music signals are usually simpler in frequency content than in time.
- e.g. The .mp3 standard.

Transfer Functions

Suppose we know the output of a system for every possible input of the form $e^{-\imath\omega t}$.

- Input $u_\omega(t) = e^{-\imath\omega t}$ produces output $g(\omega, t)$

Then for any input with Fourier Transform $\hat{u}(\imath\omega)$, we can calculate the output y as

$$y(t) = \int_{-\infty}^{\infty} \hat{u}(\imath\omega) g(\omega, t) d\omega$$

Summary:

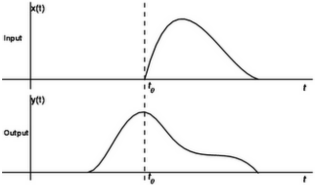
- Let $\hat{u}(\imath\omega)$ be the Fourier Transform of the input, $u(t)$.
- Let $\hat{y}(\imath\omega)$ be the Fourier Transform of the outut, $y(t)$.
- Then

  $$\hat{y}(\imath\omega) = \hat{G}(\imath\omega)\hat{u}(\imath\omega)$$

  for some Transfer Function, $\hat{G}$

Problem: Sinusoids are NOT REAL! Every signal has to START and STOP...
Next: We consider the Laplace Transform.

Causality



Definition 5.

In a **Causal** System, a change in input at a later time $u(t+\tau)$ cannot affect the present output $y(t)$.

Every physical system in the universe is causal.

Causal:

$$\dot{x}(t) = x(t)$$

Non-Causal

$$\dot{x}(t) = x(t+1)$$

Non-Causal Systems are usually artificial.

- Noise filtering
  - Averages window of future and past.
  - Playback must be delayed due to causality

Conclusion: We only consider causal systems in this class.

For causal systems, we can use the Laplace Transform:

Definition 6.

The **Laplace Transform** of the signal u can be found as

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t) dt$$

Similar to the Fourier Transform, we have the following property.

Theorem 7.

For any causal, bounded, linear, time-invariant system, there exists a function $\hat{G}(s)$ such that

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

where $\hat{y}(s)$ is the Laplace Transform of the output.
The function, $\hat{G}$ is called the **Transfer Function** *of the system.*

Summary

What have we learned today?

Simple ways to use state-space.

- How to find the output given an input.
- Linearity

Linear Systems.

- The Fourier Series
  - Representing signals as the sum of sinusoids.
  - Representing systems using response to sinusoids.
- The Fourier Transform
- The Laplace Transform
  - Representing signals in the frequency domain.
  - Representing systems using response to sinusoids.

Next Lecture: Calculating the Laplace Transform