Introduction

In this Lecture, you will learn: Transfer Functions

- Transfer Function Representation of a System
- State-Space to Transfer Function
- Direct Calculation of Transfer Functions

Block Diagram Algebra

- Modeling in the Frequency Domain
- Reducing Block Diagrams
Previously:
The Laplace Transform of a Signal

**Definition:** We defined the Laplace transform of a Signal.

- **Input**, \( \hat{u} = \mathcal{L}(u) \).
- **Output**, \( \hat{y} = \mathcal{L}(y) \)

**Theorem 1.**

*Any bounded, linear, causal, time-invariant system, \( G \), has a Transfer Function, \( \hat{G} \), so that if \( y = Gu \), then*

\[
\hat{y}(s) = \hat{G}(s)\hat{u}(s)
\]

There are several ways of finding the *Transfer Function.*
Transfer Functions
Example: Simple System

**State-Space:**

\[
\begin{align*}
\dot{x}(t) &= -x(t) + u(t) \\
y(t) &= x(t) - 0.5u(t) \quad x(0) = 0
\end{align*}
\]

Apply the Laplace transform to the first equation:

\[
\mathcal{L}\left(\dot{x}(t) = -x(t) + u(t)\right) \quad \text{which gives} \quad s\hat{x}(s) = -\hat{x}(s) + \hat{u}(s).
\]

Solving for \(\hat{x}(s)\), we get

\[
(s + 1)\hat{x}(s) = \hat{u}(s) \quad \text{and so} \quad \hat{x}(s) = \frac{1}{s + 1}\hat{u}(s).
\]

Similarly, the second equation yields:

\[
\hat{y}(s) = \hat{x}(s) - 0.5\hat{u}(s) = \frac{1}{s + 1}\hat{u}(s) - 0.5\hat{u}(s) = \frac{1 - 0.5(s + 1)}{s + 1}\hat{u}(s) = \frac{1}{2}\frac{s - 1}{s + 1}\hat{u}(s)
\]

Thus we have the **Transfer Function**:

\[
\hat{G}(s) = \frac{1}{2}\frac{s - 1}{s + 1}
\]
Transfer Functions

Example: Step Response

The *Transfer Function* provides a convenient way to find the response to inputs.

**Step Input Response:** \( \hat{u}(s) = \frac{1}{s} \)

\[
\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{2} \frac{s-1}{s+1} \frac{1}{s} = \frac{1}{2} \frac{s-1}{s^2 + s}
\]

\[
= \frac{1}{2} \left( \frac{2}{s+1} - \frac{1}{s} \right)
\]

Consulting our table of Laplace Transforms,

\[
y(t) = \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{2}{s+1} \right] - \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{1}{s} \right]
\]

\[
= e^{-t} - \frac{1}{2} 1(t)
\]
Transfer Functions
Example: Sinusoid Response

Sine Function: \( \hat{u}(s) = \frac{1}{s^2+1} \)

\[
\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{2} \frac{s - 1}{s^3 + s^2 + s + 1} = \frac{1}{2} \left( \frac{s}{s^2 + 1} - \frac{1}{s + 1} \right)
\]

Consulting our table of Laplace Transforms,

\[
y(t) = \frac{1}{2} \cos t - \frac{1}{2} e^{-t}
\]

Note that this is the same answer we got by integration in Lecture 4.
Inverted Pendulum Example

Return to the pendulum.

**Dynamics:**

\[ \ddot{\theta}(t) = \frac{Mgl}{2J} \theta(t) + \frac{1}{J} T(t) \]
\[ y(t) = \theta(t) \]

For the first equation,

\[ s^2 \hat{\theta}(s) = \frac{Mgl}{2J} \hat{\theta}(s) + \frac{1}{J} \hat{T}(s) \]

Solve for \( \hat{\theta}(s) \):

\[ \hat{\theta}(s) = \frac{1}{J} s^2 - \frac{Mgl}{2J} \hat{T}(s) \]

**Second Equation:** \( \hat{y}(s) = \hat{\theta}(s) \)

**Transfer Function:**

\[ \hat{G}(s) = \frac{\hat{y}(s)}{\hat{T}(s)} = \frac{1}{J} s^2 - \frac{Mgl}{2J} \]
**Inverted Pendulum Example: Impulse Response**

**Impulse Input:** \( \hat{u}(s) = 1 \)

\[
\hat{y}(s) = \hat{G}(s)\hat{u}(s) = \frac{1}{J} \frac{1}{s^2 - \frac{Mgl}{2J}}
\]

\[
= \frac{1}{J} \left( \frac{1}{s - \sqrt{\frac{Mgl}{2J}}} \right) \left( \frac{1}{s + \sqrt{\frac{Mgl}{2J}}} \right)
\]

\[
= \frac{1}{J} \sqrt{\frac{2J}{Mgl}} \left( \frac{1}{s - \sqrt{\frac{Mgl}{2J}}} - \frac{1}{s + \sqrt{\frac{Mgl}{2J}}} \right)
\]

**Figure:** Impulse Response with \( g = l = J = 1, \ M = 2 \)

In time-domain:

\[
y(t) = \frac{1}{J} \sqrt{\frac{2J}{Mgl}} \left( e^{\sqrt{\frac{Mgl}{2J}} t} - e^{-\sqrt{\frac{Mgl}{2J}} t} \right)
\]

Pendulum Accelerates to infinity!
Recall the dynamics:

\[
\ddot{z}_1(t) = -\frac{K_1}{m_c} z_1(t) - \frac{c}{m_c} \dot{z}_1(t) + \frac{K_1}{m_c} z_2(t) + \frac{c}{m_c} \dot{z}_2(t)
\]

\[
\ddot{z}_4(t) = \frac{K_1}{m_w} z_1(t) + \frac{c}{m_w} \dot{z}_1(t) - \left(\frac{K_1}{m_w} + \frac{K_2}{m_w}\right) z_2(t) - \frac{c}{m_w} \dot{z}_2(t) - \frac{K_2}{m_w} u(t)
\]

\[
y(t) = [z_2(t)]
\]
Apply the Laplace Transform to the dynamics:

\[ s^2 \hat{z}_1(s) = -\frac{K_1}{m_c} \hat{z}_1(s) - \frac{c}{m_c} s \hat{z}_1(s) + \frac{K_1}{m_c} \hat{z}_2(s) + \frac{c}{m_c} s \hat{z}_2(s) \]

\[ s^2 \hat{z}_2(s) = \frac{K_1}{m_w} \hat{z}_1(s) + \frac{c}{m_w} s \hat{z}_1(s) - \left( \frac{K_1}{m_w} + \frac{K_2}{m_w} \right) \hat{z}_2(s) - \frac{c}{m_w} s \hat{z}_2(s) - \frac{K_2}{m_w} \hat{u}(s) \]

\[ \hat{y}(s) = \hat{z}_2(s) \]
Constructing the Transfer Function: Suspension System

We isolate the $z_1$ and $z_2$ terms:

\[
\left(s^2 + \frac{c}{m_c}s + \frac{K_1}{m_c}\right) \hat{z}_1(s) = \left(\frac{K_1}{m_c} + \frac{c}{m_c}s\right) \hat{z}_2(s)
\]

\[
\left(s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}\right) \hat{z}_2(s) = \left(\frac{K_1}{m_w} + \frac{c}{m_w}s\right) \hat{z}_1(s) - \frac{K_2}{m_w} \hat{u}(s)
\]

\[
\hat{y}(s) = \hat{z}_2(s)
\]

Which yields

\[
\hat{z}_1(s) = \frac{\left(\frac{K_1}{m_c} + \frac{c}{m_c}s\right)}{\left(s^2 + \frac{c}{m_c}s + \frac{K_1}{m_c}\right)} \hat{z}_2(s)
\]

\[
\hat{z}_2(s) = \frac{\frac{K_1}{m_w} + \frac{c}{m_w}s}{s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}} \hat{z}_1(s) - \frac{\frac{K_2}{m_w}}{s^2 + \frac{c}{m_w}s + \frac{K_1}{m_w} + \frac{K_2}{m_w}} \hat{u}(s)
\]
Constructing the Transfer Function: Suspension System

Now we can plug in for \( \hat{z}_1 \) and solve for \( \hat{z}_2 \):

\[
\hat{z}_2(s) = \frac{K_2(m_cs^2 + cs + K_1)}{m_cm/ws^4 + c(m_w + mc)s^3 + (K_1mc + K_1m_w + K_2mc)s^2 + cK_2s + K_1K_2} \hat{u}(s)
\]

Compare to the State-Space Representation:

\[
\frac{d}{dt} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{mc} & -\frac{c}{mc} & \frac{K_1}{mc} & \frac{c}{mc} \\ \frac{K_1}{m_w} & \frac{c}{m_w} & 0 & 1 \\ \frac{K_1}{m_w} & \frac{c}{m_w} & -\left(\frac{K_1}{m_w} + \frac{K_2}{m_w}\right) & -\frac{c}{m_w} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{K_2}{m_w} \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t)
\]
The interconnection of systems can be represented by block diagrams.

**Cascade of Systems:** Suppose we have two systems: $G$ and $H$.

**Definition 2.**

The **Cascade** or **Series** interconnection of two systems is:

$$y_1 = Gu \quad y = Hy_1$$

or

$$y = H(G(u))$$
Series Interconnection:

- The output of $G$ is the input to $H$.
- Let $\hat{G}(s)$ and $\hat{H}(s)$ be the transfer functions for $G$ and $H$.
- Then

\[ \hat{y}_1(s) = \hat{G}_1(s)\hat{u}(s) \quad \hat{y}(s) = \hat{H}(s)\hat{y}_1(s) = \hat{H}(s)\hat{G}(s)\hat{u}(s) \]

- The Transfer Function, $\hat{T}(s)$ for the combination of $G$ and $H$ is

\[ \hat{T}(s) = \hat{H}(s)\hat{G}(s) \]

Note: The order of the $\hat{G}$ and $\hat{H}$!
Parallel Interconnection: Suppose we have two systems: $G$ and $H$.

**Definition 3.**

The Parallel interconnection of two systems is

$$y_1 = G u \quad y_2 = H u \quad y = y_1 + y_2$$

or

$$y = H(u) + G(u)$$
The Transfer function of a Parallel interconnection:

• Laplace transform:

\[
\hat{y}(s) = \hat{y}_1(s) + \hat{y}_2(s) = \hat{G}(s)\hat{u}(s) + \hat{H}(s)\hat{u}(s) = \left(\hat{H}(s) + \hat{G}(s)\right)\hat{u}(s)
\]

• The *Transfer Function*, \(\hat{T}(s)\) for the parallel interconnection of \(G\) and \(H\) is

\[
\hat{T}(s) = \hat{H}(s) + \hat{G}(s)
\]
Feedback:

- **Controller:** $z = K(u - y)$  
  **Plant:** $y = Gz$

In the Frequency Domain:

$$\hat{z}(s) = -\hat{K}(s)\hat{y}(s) + \hat{K}(s)\hat{u}(s)$$  
$$\hat{y}(s) = \hat{G}(s)\hat{z}(s)$$

so

$$\hat{y}(s) = \hat{G}(s)\hat{z}(s) = -\hat{G}(s)\hat{K}(s)\hat{y}(s) + \hat{G}(s)\hat{K}(s)\hat{u}(s)$$

Solving for $\hat{y}(s)$,

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$
There is another Feedback interconnection

- $u$ is the input
- $y$ is the output

\[ \hat{y}(s) = \hat{G}(s)\hat{z}(s) \]
\[ \hat{z}(s) = u(s) - \hat{K}(s)\hat{y}(s) \]

Which yields

\[ \hat{y}(s) = \hat{G}(s) \left( u(s) - \hat{K}(s)\hat{y}(s) \right) = \hat{G}(s)\hat{u}(s) - \hat{G}(s)\hat{K}(s)\hat{y}(s) \]

hence the Transfer Function is:

\[ \hat{y}(s) = \frac{\hat{G}(s)}{1 + \hat{G}(s)\hat{K}(s)} \hat{u}(s). \]
The Effect of Feedback: Impulse Response
Inverted Pendulum Model

Transfer Function
\[ \hat{G}(s) = \frac{1}{J s^2 - \frac{Mgl}{2}} \]

Controller: Static Gain: \( \hat{K}(s) = K \)
Input: Impulse: \( \hat{u}(s) = 1 \).
Closed Loop: Lower Feedback

\[ \hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)} \hat{u}(s) = \frac{K}{J s^2 - \frac{Mgl}{2}} = \frac{\hat{y}(s)}{1 + \hat{G}(s)\hat{K}(s)} = \frac{K}{J s^2 - \frac{Mgl}{2} + K} \]

First Case:
- If \( K > \frac{Mgl}{2} \), then \( K - \frac{Mgl}{2} > 0 \), so

\[ \hat{y}(s) = \frac{K}{s^2 + \left( \frac{K}{J} - \frac{Mgl}{2J} \right)} \]

\[ y(t) = \frac{K}{J \sqrt{\frac{K}{J} - \frac{Mgl}{2J}}} \sin \left( \sqrt{\frac{K}{J} - \frac{Mgl}{2J}} t \right) \]
Second Case:

- If \( K < \frac{Mgl}{2} \), then \( K - \frac{Mgl}{2} < 0 \), so

\[
\hat{y}(s) = \frac{K}{J} \left( \frac{1}{s - \sqrt{K/J - \frac{Mgl}{2J}}} + \frac{1}{s + \sqrt{K/J - \frac{Mgl}{2J}}} \right)
\]

\[
y(t) = \frac{K}{J} \left( e^{\sqrt{K/J - \frac{Mgl}{2J}} t} + e^{-\sqrt{K/J - \frac{Mgl}{2J}} t} \right)
\]

**Important:** Value of \( K \) determines stability vs. instability
Now let's look at how to reduce a more complicated interconnections.

Label

- The output from the inner loop $z$
- The input to the inner loop $u$

First **Close the Inner Loop** using the *Lower Feedback Interconnection*.

$$\hat{z}(s) = \frac{K_1}{s + 1} \hat{u}(s) = \frac{K_1}{K_1 + s} \hat{u}(s)$$
We now have a reduced Block Diagram

\[
\hat{y}(s) = \frac{K_1}{s(K_1+s)} \hat{e}(s) = \frac{K_1}{s(K_1+s) + K_1} \hat{e}(s)
\]

So the Transfer function is \( \hat{T}(s) = \frac{K_1}{s^2 + K_1 s + K_1} \)
Summary

What have we learned today?

Transfer Functions
- Transfer Function Representation of a System
- State-Space to Transfer Function
- Direct Calculation of Transfer Functions

Block Diagram Algebra
- Modeling in the Frequency Domain
- Reducing Block Diagrams

Next Lecture: Partial Fraction Expansion