Partial Fractions

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Lecture 7: The Partial Fraction Expansion

Introduction

In this Lecture, you will learn: The Inverse Laplace Transform

- Simple Forms

The Partial Fraction Expansion

- How poles relate to dominant modes
- Expansion using single poles
- Repeated Poles
- Complex Pairs of Poles
    - Inverse Laplace

Recall: The Inverse Laplace Transform of a Signal

To go from a frequency domain signal, $\hat{u}(s)$, to the time-domain signal, $u(t)$, we use the Inverse Laplace Transform.

Definition 1.

The **Inverse Laplace Transform** of a signal $\hat{u}(s)$ is denoted $u(t) = \mathcal{L}^{-1}\hat{u}$.

$$u(t) = \mathcal{L}^{-1}\hat{u} = \int_0^\infty e^{\imath\omega t}\hat{u}(\imath\omega)d\omega$$

- Like $\mathcal{L}$, the inverse Laplace Transform $\mathcal{L}^{-1}$ is also a Linear system.
- **Identity:** $\mathcal{L}^{-1}\mathcal{L}u = u$.
- Calculating the Inverse Laplace Transform can be tricky. e.g.

$$\hat{y} = \frac{s^3 + s^2 + 2s - 1}{s^4 + 3s^3 - 2s^2 + s + 1}$$

Poles and Rational Functions

Definition 2.

A **Rational Function** is the ratio of two polynomials:

$$\hat{u}(s) = \frac{n(s)}{d(s)}$$

Most transfer functions are rational.

Definition 3.

The point s_p is a **Pole** of the rational function $\hat{u}(s) = \frac{n(s)}{d(s)}$ if $d(s_p) = 0$.

- It is convenient to write a rational function using its poles

$$\frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- The Inverse Laplace Transform of an isolated pole is easy:

$$\hat{u}(s) = \frac{1}{s+p} \qquad \text{means} \qquad u(t) = e^{-pt}$$

Partial Fraction Expansion

Definition 4.

The **Degree** of a polynomial $n(s)$, is the highest power of s with a nonzero coefficient.

Example: The degree of $n(s)$ is 4

$$n(s) = s^4 + .5s^2 + 1$$

Definition 5.

A rational function $\hat{u}(s) = \frac{n(s)}{d(s)}$ is **Strictly Proper** if the degree of $n(s)$ is less than the degree of $d(s)$.

- We assume that $n(s)$ has lower degree than $d(s)$
- Otherwise, perform long division until we have a **strictly proper** remainder

$$\frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} = s + 1 + \frac{2}{s^2 + s + 5}$$

Poles and Inverse Laplace Transforms

A Strictly Proper Rational Function is The Sum of Poles: We can *usually* find coefficients r_i such that

$$\frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{r_1}{s-p_1} + \ldots + \frac{r_n}{s-p_n}$$

- Except in the case of repeated poles.

Poles Dominate the Motion: Because a signal is the sum of poles, The inverse Laplace has the form

$$u(t) = r_1 e^{p_1 t} + \ldots + r_n e^{p_n t}$$

- p_i may be complex.
  - If p_i are complex, r_i may be complex.
- Doesn't hold for repeated poles.

Examples

Simple State-Space: Step Response

$$\hat{y}(s) = \frac{s-1}{(s+1)s} = \frac{2}{s+1} - \frac{1}{s}$$

$$y(t) = e^{-t} - \frac{1}{2}\mathbf{1}(t)$$

Suspension System: Impulse Response

$$\hat{y}(s) = \frac{1}{J}\frac{1}{s^2 - \frac{Mgl}{2J}} = \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(\frac{1}{s-\sqrt{\frac{Mgl}{2J}}} - \frac{1}{s+\sqrt{\frac{Mgl}{2J}}}\right)$$

$$y(t) = \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(e^{\sqrt{\frac{Mgl}{2J}}t} - e^{-\sqrt{\frac{Mgl}{2J}}t}\right)$$

Simple State-Space: Sinusoid Response

$$\hat{y}(s) = \frac{s-1}{(s+1)(s^2+1)} = \left(\frac{s}{s^2+1} - \frac{1}{s+1}\right)$$

$$y(t) = \frac{1}{2}\cos t - \frac{1}{2}e^{-t}$$

Partial Fraction Expansion

Conclusion: If we can find coefficients r_i such that

$$\hat{u}(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$
$$= \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n},$$

then this is the **PARTIAL FRACTION EXPANSION** of $\hat{u}(s)$.

We will address several cases of increasing complexity:

1. $d(s)$ has all real, non-repeating roots.
2. $d(s)$ has all real roots, some repeating.
3. $d(s)$ has complex, repeated roots.

Case 1: Real, Non-repeated Roots

This case is the easiest:

$$\hat{y}(s) = \frac{n(s)}{(s - p_1) \cdots (s - p_n)}$$

where p_n are all real and distinct.

Theorem 6.

For case 1, $\hat{y}(s)$ can always be written as

$$\hat{y}(s) = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

where the r_i are all real constants

The trick is to solve for the r_i

- There are n unknowns - the r_i.
- To solve for the r_i, we evaluate the equation for values of s.
  - Potentially unlimited equations.
  - We only need n equations.
  - We will evaluate at the points $s = p_i$.

Case 1: Real, Non-repeated Roots

Solving

We want to solve

$$\hat{y}(s) = \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n} \qquad \text{for } r_1,\ r_2,\ \ldots$$

For each r_i, we Multiply by $(s - p_i)$.

$$\begin{aligned}\hat{y}(s)(s-p_i) &= r_1 \frac{s-p_i}{s-p_1} + \cdots + r_i \frac{s-p_i}{s-p_i} + \cdots + r_n \frac{s-p_i}{s-p_n} \\ &= r_1 \frac{s-p_i}{s-p_1} + \cdots + r_i + \cdots + r_n \frac{s-p_i}{s-p_n}\end{aligned}$$

Evaluate the Right Hand Side at $s = p_i$: We get

$$\begin{aligned}&r_1 \frac{p_i-p_i}{p_i-p_1} + \cdots + r_i + \cdots + r_n \frac{p_i-p_i}{p_i-p_n} \\ &= r_1 \frac{0}{p_i-p_1} + \cdots + r_i + \cdots + r_n \frac{0}{p_i-p_n} = r_i\end{aligned}$$

Setting $LHS = RHS$, we get a simple formula for r_i:

$$r_i = \hat{y}(s)(s-p_i)|_{s=p_i}$$

Case 1: Real, Non-repeated Roots

Calculating r_i

So we can find all the coefficients:

$$r_i = \hat{y}(s)(s - p_i)|_{s=p_i}$$

$$\begin{aligned}\hat{y}(s)(s-p_i) &= \frac{n(s)}{(s-p_1)\cdots(s-p_{i-1})(s-p_i)(s-p_{i+1})\cdots(s-p_n)}(s-p_i) \\ &= \frac{n(s)}{(s-p_1)\cdots(s-p_{i-1})(s-p_{i+1})\cdots(s-p_n)}\end{aligned}$$

- This is $\hat{y}$ with the pole $s - p_i$ removed.

Definition 7.

The **Residue** of $\hat{y}$ at $s = p_i$ is the value of $\hat{y}(p_i)$ with the pole $s - p_i$ removed.

Thus we have

$$r_i = \hat{y}(s)(s - p_i)|_{s=p_i} = \frac{n(p_i)}{(p_i-p_1)\cdots(p_i-p_{i-1})(p_i-p_{i+1})\cdots(p_i-p_n)}$$

Example: Case 1

$$\hat{y}(s) = \frac{2s+6}{(s+1)(s+2)}$$

We first separate $\hat{y}$ into parts:

$$\hat{y}(s) = \frac{2s+6}{(s+1)(s+2)} = \frac{r_1}{s+1} + \frac{r_2}{s+2}$$

Residue 1: we calculate:

$$r_1 = \frac{2s+6}{(s+1)(s+2)}(s+1)|_{s=-1} = \frac{2s+6}{s+2}|_{s=-1} = \frac{-2+6}{-1+2} = \frac{4}{1} = 4$$

Residue 2: we calculate:

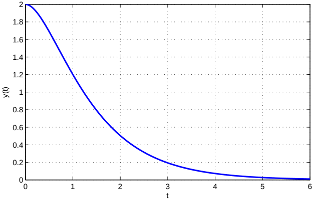
$$r_2 = \frac{2s+6}{(s+1)(s+2)}(s+2)|_{s=-2} = \frac{2s+6}{s+1}|_{s=-2} = \frac{-4+6}{-2+1} = \frac{2}{-1} = -2$$

Thus

$$\hat{y}(s) = \frac{4}{s+1} + \frac{-2}{s+2}$$

Concluding,

$$y(t) = 4e^{-t} - 2e^{-2t}$$



Case 2: Real, Repeated Roots

Sometimes $\hat{y}$ has a repeated pole:

$$\hat{y}(s) = \frac{n(s)}{(s-p_1)^q(s-p_2)\cdots(s-p_n)}$$

In this case, we **CAN NOT** use simple expansion (As in case 1). Instead we have

$$\begin{aligned}\hat{y}(s) &= \frac{n(s)}{(s-p_1)^q(s-p_2)\cdots(s-p_n)} \\ &= \left(\frac{r_{11}}{(s-p_1)} + \frac{r_{12}}{(s-p_1)^2} + \cdots + \frac{r_{1q}}{(s-p_1)^q}\right) + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}\end{aligned}$$

- The r_{ij} are still real-valued.

Example: Find the r_{ij}

$$\frac{(s+3)^2}{(s+2)^3(s+1)^2} = \frac{r_{11}}{s+2} + \frac{r_{12}}{(s+2)^2} + \frac{r_{13}}{(s+2)^3} + \frac{r_{21}}{s+1} + \frac{r_{22}}{(s+1)^2}$$

Case 2: Real, Repeated Roots

Problem: We have more coefficients to find.

- q coefficients for each repeated root $(s - p_i)^q$.

First Step: Solve for $r_2, \cdots, r_n$ as before:
If p_i is not a repeated root, then

$$r_i = \frac{n(p_i)}{(p_i - p_1)^q \cdots (p_i - p_{i-1})(p_i - p_{i+1}) \cdots (p_i - p_n)}$$

Case 2: Real, Repeated Roots

New Step: Multiply by $(s-p_1)^q$ to get the coefficient r_{1q}.

$$\begin{aligned}\hat{y}(s)(s-p_1)^q &= \left(r_{11}\frac{(s-p_1)^q}{(s-p_1)} + r_{12}\frac{(s-p_1)^q}{(s-p_1)^2} + \cdots + r_{1q}\frac{(s-p_1)^q}{(s-p_1)^q}\right) \\ &\quad + r_2\frac{(s-p_1)^q}{s-p_2} + \cdots + r_n\frac{(s-p_1)^q}{s-p_n} \\ &= \left(r_{11}(s-p_1)^{q-1} + r_{12}(s-p_1)^{q-2} + \cdots + r_{1q}\right) \\ &\quad + r_2\frac{(s-p_1)^q}{s-p_2} + \cdots + r_n\frac{(s-p_1)^q}{s-p_n}\end{aligned}$$

and evaluate at the point $s=p_i$ to get .

$$r_{1q} = \hat{y}(s)(s-p_1)^q|_{s=p_1}$$

- r_{1q} is $\hat{y}(p_1)$ with the repeated pole removed.

Case 2: Real, Repeated Roots

To find the remaining coefficients, we Differentiate:

$$\frac{d}{ds}\left(\hat{y}(s)(s-p_1)^q\right) = \frac{d}{ds}\Bigg(\big(r_{11}(s-p_1)^{q-1} + r_{12}(s-p_1)^{q-2} + \cdots + r_{1q}\big) + r_2\frac{(s-p_1)^q}{s-p_2} + \cdots + r_n\frac{(s-p_1)^q}{s-p_n}\Bigg)$$

to get

$$\begin{aligned}\frac{d}{ds}\left(\hat{y}(s)(s-p_1)^q\right) &= \big((q-1)r_{11}(s-p_1)^{q-2} + (q-2)r_{12}(s-p_1)^{q-3} + \cdots + r_{1,(q-1)}\big) \\ &\quad + \frac{d}{ds}(s-p_1)^q\left(\frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}\right)\end{aligned}$$

Evaluating at the point $s = p_1$.

$$\frac{d}{ds}\left(\hat{y}(s)(s-p_1)^q\right)|_{s=p_1} = r_{1,(q-1)}$$

Case 2: Real, Repeated Roots

If we differentiate again, we get

$$\begin{aligned}\frac{d^2}{ds^2}\left(\hat{y}(s)(s-p_1)^q\right) \\ &= \left((q-1)(q-2)r_{11}(s-p_1)^{q-3} + \cdots + 2r_{1,(q-2)}\right) \\ &\quad + \frac{d^2}{ds^2}(s-p_1)^q\left(\frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}\right)\end{aligned}$$

Evaluating at $s-p_1$, we get

$$r_{1,(q-2)} = \frac{1}{2}\frac{d^2}{ds^2}\left(\hat{y}(s)(s-p_1)^q\right)|_{s=p_1}$$

Extending this indefinitely

$$r_{1,j} = \frac{1}{(q-j-1)!}\frac{d^{q-j-1}}{ds^{q-j-1}}\left(\hat{y}(s)(s-p_1)^q\right)|_{s=p_1}$$

Of course, calculating this derivative is often difficult.

Example: Real, Repeated Roots

Expand a simple example

$$\hat{y}(s) = \frac{s+3}{(s+2)^2(s+1)} = \frac{r_{11}}{s+2} + \frac{r_{12}}{(s+2)^2} + \frac{r_2}{s+1}$$

First Step: Calculate r_2

$$r_2 = \frac{s+3}{(s+2)^2}|_{s=-1} = \frac{-1+3}{(-1+2)^2} = \frac{2}{1} = 2$$

Second Step: Calculate r_{12}

$$r_{12} = \frac{s+3}{s+1}|_{s=-2} = \frac{-2+3}{-2+1} = \frac{1}{-1} = -1$$

The Difficult Step: Calculate r_{11}:

$$\begin{aligned} r_{11} &= \frac{d}{ds}\left(\frac{s+3}{s+1}\right)|_{s=-2} \\ &= \left(\frac{1}{s+1} - \frac{s+3}{(s+1)^2}\right)|_{s=-2} = \frac{1}{-2+1} - \frac{-2+3}{(-2+1)^2} = \frac{1}{-1} - \frac{1}{1} = -2 \end{aligned}$$

Example: Real, Repeated Roots

So now we have

$$\hat{y}(s) = \frac{s+3}{(s+2)^2(s+1)} = \frac{-2}{s+2} + \frac{-1}{(s+2)^2} + \frac{2}{s+1}$$

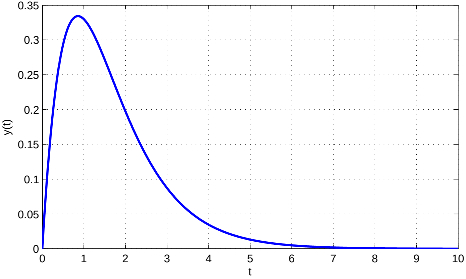
Question: what is the Inverse Fourier Transform of $\frac{1}{(s+2)^2}$?

Recall the Power Exponential:

$$\frac{1}{(s+a)^m} \to \frac{t^{m-1}e^{-at}}{(m-1)!}$$

Finally, we have:

$$y(t) = -2e^{-2t} - te^{-2t} + 2e^{-t}$$



Case 3: Complex Roots

Most signals have complex roots (More common than repeated roots).

$$\hat{y}(s) = \frac{3}{s(s^2 + 2s + 5)}$$

has roots at $s = 0$, $s = -1 - 2\imath$, and $s = -1 + 2\imath$.

Note that:

- Complex roots come in pairs.
- Simple partial fractions will work, but NOT RECOMMENDED
  - Coefficients will be complex.
  - Solutions will be complex exponentials.
  - Require conversion to real functions.

Best to Separate out the Complex pairs as:

$$\hat{y}(s) = \frac{n(s)}{(s^2 + as + b)(s - p_1) \cdots (s - p_n)}$$

Case 3: Complex Roots

Complex pairs have the expansion

$$\frac{n(s)}{(s^2+as+b)(s-p_1)\cdots(s-p_n)} = \frac{k_1 s + k_2}{s^2+as+b} + \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n}$$

Note that there are two coefficients for each pair: k_1 and k_2.

NOTE: There are several different methods for finding k_1 and k_2.

First Step: Solve for $r_2, \cdots, r_n$ as normal.

Second Step: Clear the denominator. Multiply equation by all poles.

$$n(s) = (s^2+as+b)(s-p_1)\cdots(s-p_n)\left(\frac{k_1 s + k_2}{s^2+as+b} + \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n}\right)$$

Third Step: Solve for k_1 and k_2 by examining the coefficients of powers of s.

Warning: May get complicated or impossible for multiple complex pairs.

Case 3 Example

Take the example

$$\hat{y}(s) = \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{k_1 s + k_2}{s^2+4} + \frac{r_1}{s+1}$$

First Step: Find the simple coefficient r_1.

$$r_1 = \frac{2(s+2)}{s^2+4}|_{s=-1} = 2\frac{-1+2}{-1^2+4} = 2\frac{1}{5} = \frac{2}{5}$$

Next Step: Multiply through by $(s^2+4)(s+1)$.

$$2(s+2) = (s+1)(k_1 s + k_2) + r_1(s^2+4)$$

Expanding and using $r_1 = 2/5$ gives:

$$2s + 4 = (k_1 + 2/5)s^2 + (k_2 + k_1)s + k_2 + 8/5$$

Case 3 Example

Recall the equation

$$2s + 4 = (k_1 + 2/5)s^2 + (k_2 + k_1)s + k_2 + 8/5$$

Equating coefficients gives 3 equations:

- s^2 **term** - $0 = k_1 + 2/5$
- s **term** - $2 = k_2 + k_1$
- s^0 **term** - $4 = k_2 + 8/5$

An over-determined system of equations (but consistent)

- First term gives $k_1 = -2/5$
- Second term gives $k_2 = 2 - k_1 = 10/5 + 2/5 = 12/5$
- Double-Check with last term: $4 = 12/5 + 8/5 = 20/5$

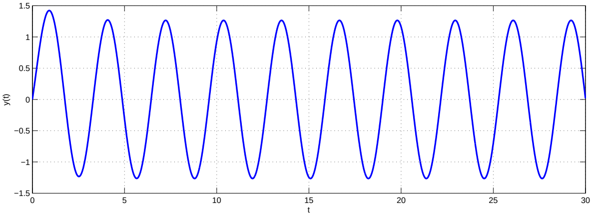
Case 3 Example

So we have

$$\hat{y}(s) = \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{2}{5}\left(\frac{1}{s+1} - \frac{s-6}{s^2+4}\right)$$

$$= \frac{2}{5}\left(\frac{1}{s+1} - \frac{s}{s^2+4} + \frac{6}{s^2+4}\right)$$

$$y(t) = \frac{2}{5}\left(e^{-t} - \cos(2t) + 3\sin(2t)\right)$$



Case 3 Example

Now consider the solution to **A Different Numerical Example:**

$$\frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5}$$

What to do with term:

$$\frac{s+2}{s^2+2s+5}?$$

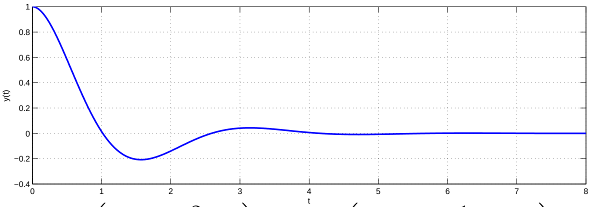
We can rewrite as the combination of a frequency shift and a sinusoid:

$$\begin{aligned}\frac{s+2}{s^2+2s+5} &= \frac{s+2}{s^2+2s+1+4} = \frac{(s+1)+1}{(s+1)^2+4} \\ &= \frac{(s+1)}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}\end{aligned}$$

Case 3 Example

$$\frac{s+2}{s^2+2s+5} = \frac{(s+1)}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}$$

- $\frac{(s+1)}{(s+1)^2+4}$ is the sinusoid $\frac{s}{s^2+4}$ shifted by $s \to s+1$.
- $s \to s+1$ means multiplication by e^{-t} in the time-domain.
- $$\mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+4}\right) = e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = e^{-t}\cos 2t$$
- Likewise $\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) = e^{-t}\sin 2t$



$$\mathcal{L}^{-1}\left(\frac{s+2}{s^2+2s+5}\right) = e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Summary

What have we learned today?

In this Lecture, you will learn: The Inverse Laplace Transform

- Simple Forms

The Partial Fraction Expansion

- How poles relate to dominant modes
- Expansion using single poles
- Repeated Poles
- Complex Pairs of Poles
  - Inverse Laplace

Next Lecture: Important Properties of the Response