Systems Analysis and Control

Matthew M. Peet
Arizona State University

Lecture 8: Response Characteristics
Overview

In this Lecture, you will learn:

The Effects of Feedback on Dynamic Response
- Changes in Transfer Function
- Stability
- Impact of Poles on Dynamic Response

Real Poles
- Steady-State Error
- Rise Time
- Settling Time

Complex Poles
- Complex Pole Locations
- Damped/Natural Frequency
- Damping and Damping Ratio
The Effect of Feedback Control

Recall the Upper Feedback Interconnection

\[ G(s)K(s) + \]

\[ y(s) \quad u(s) \]

Feedback:

- **Controller:** \( \hat{u}(s) = K(s)(\hat{u}(s) - \hat{y}(s)) \)
- **Plant:** \( \hat{y}(s) = \hat{G}(s)\hat{u}(s) \)

The output signal is \( \hat{y}(s) \),

\[
\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)} \hat{u}(s)
\]
Controlling the Inverted Pendulum Model

Open Loop Transfer Function

$$\hat{G}(s) = \frac{1}{J s^2 - \frac{M g l}{2}}$$

Controller: Static Gain: $\hat{K}(s) = K$

Input: Impulse: $\hat{u}(s) = 1$.

Closed Loop: Lower Feedback Interconnection

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)} \hat{u}(s) = \frac{K}{1 + \frac{K}{J s^2 - \frac{M g l}{2}}} = \frac{K}{J s^2 - \frac{M g l}{2} + K}$$
Effect of Feedback on the Inverted Pendulum Model

Closed Loop Impulse Response:
Lower Feedback Interconnection

\[ \hat{y}(s) = \frac{K}{s^2 - \frac{Mgl}{2J} + \frac{K}{J}} \]

Traits:
- Infinite Oscillations
- Oscillates about 0.

Open Loop Impulse Response:

\[ \hat{y}(s) = \frac{1}{J} \sqrt{\frac{2J}{Mgl}} \left( \frac{1}{s - \sqrt{\frac{Mgl}{2J}}} - \frac{1}{s + \sqrt{\frac{Mgl}{2J}}} \right) \]

Unstable!
Controlling the Suspension System

**Open Loop Transfer Function:**
Set \( m_c = m_w = g = c = K_1 = K_2 = 1 \).

\[
\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}
\]

**Controller:** Static Gain: \( \hat{K}(s) = k \)

**Closed Loop: Lower Feedback Interconnection:**

\[
\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)
= \frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3+k)s^2 + (1+k)s + (1+k)}
\]
Controlling the Suspension Problem

Effect of changing the Feedback, $k$

**Closed Loop Step Response:**

\[
\hat{y}(s) = \frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3 + k)s^2 + (1 + k)s + (1 + k)} \frac{1}{s}
\]

High $k$:
- Overshot the target
- Quick Response
- Closer to desired value of $f$

Low $k$:
- Slow Response
- No overshoot
- Final value is farther from 1.

Questions:
- Which Traits are important?
- How to predict the behaviour?

**Figure:** Step Response for different $k$
The most basic property is **Stability**:

**Definition 1.**
A system, $G$ is **Stable** if there exists a $K > 0$ such that

$$\|Gu\|_{L^2} \leq K\|u\|_{L^2}$$

**Note:** Although this is the true definition for systems defined by transfer functions, it is rarely used.

- Bounded input means bounded output.
- Stable means $y(t) \to 0$ when $u(t) \to 0$. 

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Stability
**Definition 2.**

The **Closed Right Half-Plane**, \( CRHP \) is the set of complex numbers with non-negative real part.

\[
\{ s \in \mathbb{C} : \text{Real}(s) \geq 0 \}
\]

**Theorem 3.**

A system \( G \) is stable if and only if it’s transfer function \( \hat{G} \) has no poles in the **Closed Right Half Plane**.

- Check stability by checking poles.
- \( x \) is a pole
- \( o \) is a zero

*Figure: Unstable*
**Definition 4.**

**Steady-State Error** for a stable system is the final difference between input and output.

\[ e_{ss} = \lim_{t \to \infty} u(t) - y(t) \]

- Usually measured using the step response.
  - Since \( u(t) = 1 \),
  \[ e_{ss} = 1 - \lim_{t \to \infty} y(t) \]

**Figure:** Suspension Response for \( k = 1 \)
Recall: For any system $G$, by partial fraction expansion:

$$
\hat{y}(s) = \hat{G}(s) \frac{1}{s} = \frac{r_0}{s} + \frac{r_1}{s - p_1} + \ldots + \frac{r_n}{s - p_n}
$$

So

$$
y(t) = r_0 \mathbf{1}(t) + r_1 e^{p_1 t} + \ldots + r_n e^{p_n t}
$$

which means

$$
\lim_{t \to \infty} y(t) = r_0
$$

and hence

$$
e_{ss} = 1 - r_0
$$
The Final Value Theorem

The steady-state error is given by \( r_0 \).

\[
e_{ss} = 1 - r_0
\]

Recall: The residue at \( s = 0 \) is \( r_0 \) and is found as

\[
r_0 = \hat{G}(s)|_{s=0} = \lim_{s\to 0} \hat{G}(s)
\]

Thus the steady-state error is

\[
e_{ss} = 1 - \lim_{s\to 0} \hat{G}(s)
\]

This can be generalized to find the limit of any signal:

**Theorem 5 (Final Value Theorem).**

\[
\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s)
\]

- Assumes the limit exists (Stability)
- Can be used to find response to other inputs
  - Ramp, impulse, etc.
The Final Value Theorem for Systems in Lower Feedback

**Lower Feedback Interconnection:**

\[ \hat{y}(s) = \frac{G(s)K}{1 + G(s)K} \hat{u}(s) \]

**Error Response for Lower Feedback Interconnection:**

\[ \hat{e}(s) = \hat{u}(s) - \hat{y}(s) = \frac{1}{1 + G(s)K} \hat{u}(s) - \frac{G(s)K}{1 + G(s)K} \hat{u}(s) = \frac{1 + G(s)K - G(s)K}{1 + G(s)K} \hat{u}(s) = \frac{1}{1 + G(s)K} \hat{u}(s) \]

So the steady-state error is

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\hat{e}(s) = \lim_{s \to 0} \frac{s}{1 + G(s)K} \hat{u}(s) \]

**Error in Step Response:** If \( \hat{u}(s) = \frac{1}{s} \), then

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{1 + G(s)K} \]

**Conclusion:** Increasing \( K \) always reduces steady-state error to step!
Upper Feedback Interconnection:

\[ \hat{y}(s) = \frac{G(s)}{1 + G(s)K} \hat{u}(s) \]

Error Response for Upper Feedback Interconnection:

\[ \hat{e}(s) = \hat{u}(s) - \hat{y}(s) \]
\[ = \frac{1 + G(s)K}{1 + G'(s)K} \hat{u}(s) - \frac{G(s)}{1 + G(s)K} \hat{u}(s) \]
\[ = \frac{1 + G'(s)(K - 1)}{1 + G(s)K} \hat{u}(s) \]

Error in Step Response: If \( \hat{u}(s) = \frac{1}{s} \), then

\[ \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1 + G(s)(K - 1)}{1 + G(s)K} \]

Conclusion: Increasing \( K \) doesn’t help with step response!
The steady-state response is

\[
y_{ss} = \lim_{s \to 0} s \hat{y}(s) = \lim_{s \to 0} \hat{G}(s)
\]

\[
= \lim_{s \to 0} \frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3 + k)s^2 + (1 + k)s + (1 + k)}
\]

\[
= \frac{k}{1 + k}
\]

The steady-state error is

\[
e_{ss} = 1 - y_{ss} = 1 - \frac{k}{1 + k}
\]

\[
= \frac{1}{1 + k}
\]

- When \( k = 0 \), \( e_{ss} = 1 \)
- As \( k \to \infty \), \( e_{ss} = 0 \)
Ramp Response for Systems in Lower Feedback

**Lower Feedback Interconnection:** Recall the steady-state error is

\[
\lim_{t \to \infty} e(t) = \lim_{s \to 0} s \hat{e}(s) = \lim_{s \to 0} \frac{s}{1 + G(s)K} \hat{u}(s)
\]

**Error in Ramp Response:** If \( \hat{u}(s) = \frac{1}{s^2} \), then

\[
\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{s(1 + G(s)K)}
\]

**Conclusion:** \( \lim_{t \to \infty} e(t) = \infty! \)

**Preview of Integral Feedback:** However, if we choose \( K \to \frac{K}{s} \)

\[
\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{s(1 + G(s)\frac{K}{s})} = \lim_{s \to 0} \frac{1}{s + G(s)K}
\]

**Conclusion:** Increasing \( K \) improves the ramp response!

- More on this Later....
Dynamic Response Characteristics
Two Types of Response

From Partial Fractions Expansion, you know that motion is determined by the poles of the Transfer Function!

\[
\frac{n(s)}{(s^2 + as + b)(s - p_1) \cdots (s - p_n)} \hat{u}(s) = \frac{k_1 s + k_2}{s^2 + as + b} \hat{u}(s) + \frac{r_1}{s - p_1} \hat{u}(s) + \cdots + \frac{r_n}{s - p_n} \hat{u}(s)
\]

- Simplify the response by considering response of each pole.

**Figure: Real Pole**

**Figure: Complex Pair of Poles**

We start with **Real Poles**
**Step Response Characteristics**

**Real Poles**

**The Solution**: Step response of a real pole.

\[
\hat{y}(s) = \frac{r}{s - p} \hat{u}(s) = \frac{r}{s - p} \frac{1}{s} = \frac{r}{p} - \frac{r}{p}
\]

\[
y(t) = \frac{r}{p} (e^{pt} - 1)
\]

- Is it stable? \( (p < 0?) \)

**Cases**: Stability or Instability?
- If \( p > 0 \), then \( \lim_{t \to \infty} y(t) \to \infty \)
- If \( p < 0 \), then \( \lim_{t \to \infty} y(t) \to -\frac{r}{p} \)

**Final Value**:

\[
y_{ss} = -\frac{r}{p}
\]

**Question**: How quickly do we reach the final value?
Step Response Characteristics

Rise Time

\[ y(t) = \frac{r}{p} (e^{pt} - 1), \quad y_{ss} = -\frac{r}{p} \]

**Definition 6.**

The rise time, \( T_r \), is the time it takes to go from .1 to .9 of the final value.

If \( y(t_1) = .1y_{ss} = -.1\frac{r}{p} \), then \( t_1 \) is:

\[-.1 = e^{pt_1} - 1 \]
\[ \ln(1 - .1) = pt_1 \]
\[ t_1 = \frac{\ln .9}{p} = \frac{.11}{-p} \]

Likewise if \( y(t_2) = -.9\frac{r}{p} \), then

\[ t_2 = \frac{\ln .1}{p} = \frac{2.31}{-p} \]

Rise time \((T_r)\) for a **Single Pole** is:

\[ T_r = t_2 - t_1 = \frac{2.31}{-p} - \frac{.11}{-p} = \frac{2.2}{-p} \]
**Definition 7.**

The **Settling Time**, $T_s$, is the time it takes to reach and stay within .99 of the final value.

**Figure:** Complex Pair of Poles

LINK: Bouncing Balls
LINK: More Bouncing Balls!
LINK: Newton’s Cradle
Step Response Characteristics

Settling Time

\[ y(t) = \frac{r}{p} \left( e^{pt} - 1 \right), \quad y_{ss} = -\frac{r}{p} \]

**Definition 8.**

The **Settling Time**, \( T_s \), is the time it takes to reach and stay within .99 of the final value.

Solve \( y(T_s) = \frac{r}{p} \left( e^{pT_s} - 1 \right) = -0.99 \frac{r}{p} \) for \( T_s \):

\[ -0.99 = e^{pT_s} - 1 \]
\[ \ln(0.01) = pT_s \]
\[ T_s = \frac{\ln 0.01}{p} = -\frac{4.6}{p} \]

The settling time for a **Single Pole** is:

\[ T_s = \frac{4.6}{-p} \]
Solution for Complex Poles

\[ \hat{y}(s) = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + \omega_d^2} \frac{1}{s} = \frac{\sigma^2 + \omega_d^2}{(s + \sigma)^2 + \omega_d^2} \frac{1}{s} = -\frac{s + 2\sigma}{(s + \sigma)^2 + \omega_d^2} + \frac{1}{s} \]

The poles are at \( s = -\sigma \pm \omega_d i \). The solution is:

\[ y(t) = 1 - e^{\sigma t} \left( \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right) \]

The result is oscillation with an **Exponential Envelope**.

- Envelope decays at rate \( \sigma \)
- Speed of oscillation is \( \omega_d \), the **Damped Frequency**
Besides \( \omega_d \), there is another way to measure oscillation.

**Definition 9.**

The **Natural Frequency** of a pole at \( p = \sigma + \omega_d \) is \( \omega_n = \sqrt{\sigma^2 + \omega_d^2} \).

- for \( \hat{y}(s) = \frac{1}{s^2 + as + b} \cdot \frac{1}{s} \), \( \omega_n = \sqrt{b} \).
- Radius of the pole in complex plane.

**Resonant Frequency.**

- Also known as resonant frequency.
Step Response Characteristics
Damping Ratio

Besides $\sigma$, there are other ways to measure damping.

**Definition 10.**

The **Damping Ratio** of a pole at $p = \sigma + j\omega$ is $\zeta = \frac{|\sigma|}{\omega_n}$.

- for $\hat{y}(s) = \frac{1}{s^2 + as + b} \frac{1}{s}$, $\zeta = \frac{a}{2\sqrt{b}}$.
- Gives the ratio by which the amplitude decreases per oscillation (almost...).
We use several adjectives to describe exponential decay:

- **Undamped**
  - Oscillation continues forever, \( \sigma = \zeta = 0 \)

- **Underdamped**
  - Oscillation continues for many cycles. \( \zeta < 1 \)

- **Critically Damped**
  - No oscillation or overshoot. \( \zeta = 1, \omega_d = 0 \)

- **Overdamped**
  - When \( \zeta > 1 \), poles are real (not complex)
Summary

What have we learned today?

Characteristics of the Response

Real Poles
  • Steady-State Error
  • Rise Time
  • Settling Time

Complex Poles
  • Complex Pole Locations
  • Damped/Natural Frequency
  • Damping and Damping Ratio

Continued in Next Lecture