Systems Analysis and Control

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Lecture 9: Dynamics of Response: Complex Poles

Overview

In this Lecture, you will learn:

Characteristics of the Response

Complex Poles

- Rise Time
- Settling Time
- Percent Overshoot

Performance Specifications

- Geometric Pole Restrictions

Complex Poles

Recall: Damping and Frequency

3 Different Forms: Each with $y_{ss} = 1$

$$\hat{G}(s) = \frac{b}{s^2 + as + b}$$

$$\hat{G}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\hat{G}(s) = \frac{\sigma^2 + \omega_d^2}{s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)} = \frac{\sigma^2 + \omega_d^2}{(s - \sigma + \imath\omega_d)(s - \sigma - \imath\omega_d)} = \frac{\sigma^2 + \omega_d^2}{(s - \sigma)^2 + \omega_d^2}$$

Two Complex Poles at $s = \sigma \pm \imath\omega_d$.

- **Damped Frequency:** ω_d
  - $\omega_d = \sqrt{b - \frac{a^2}{4}} = \omega_n \sqrt{1 - \zeta^2}$
- **Decay Rate:** σ
  - $\sigma = -\frac{a}{2} = -\zeta\omega_n$
- **Natural Frequency:** ω_n
  - $\omega_n = \sqrt{b} = \sqrt{\sigma^2 + \omega_d^2}$
- **Damping Ratio:** ζ
  - $\zeta = \frac{a}{2\omega_n} = \frac{|\sigma|}{\omega_n}$

Step Response for Complex Poles

$$\hat{y}(s) = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + \omega_d^2 + \sigma^2}\frac{1}{s} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = \frac{k_1 s + k_2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{r_2}{s}$$

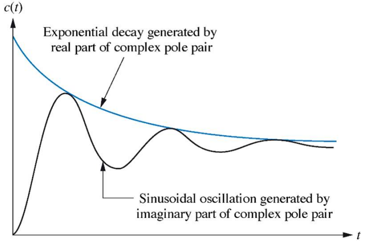
The poles of $\hat{y}$ are at $s = \sigma \pm \omega_d \imath$. Using PFE, the solution is:

$$\begin{aligned} y(t) &= 1 - e^{\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \\ &= 1 - e^{\sigma t}\frac{\omega_n}{\omega_d}\sin\left(\omega_d t + \phi\right) \end{aligned}$$

Where $\sigma = \zeta\omega_n$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$.

The result is oscillation with an Exponential Envelope.

- Envelope decays at rate σ
- Speed of oscillation is ω_d, the **Damped Frequency**



Complex Poles

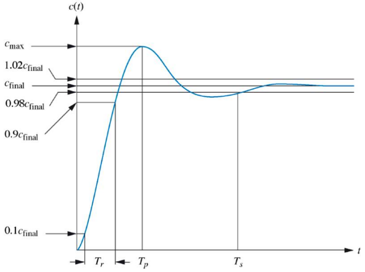
How NOT to calculate Rise Time (T_r)

Recall:

- T_r is the time to go from $.1$ to $.9$ *of the final value*.

Suppose there were no damping ($\sigma = 0$). Then the normalized solution is

$$y(t) = 1 - \cos(\omega_d t)$$



The points t_1 and t_2 occur at

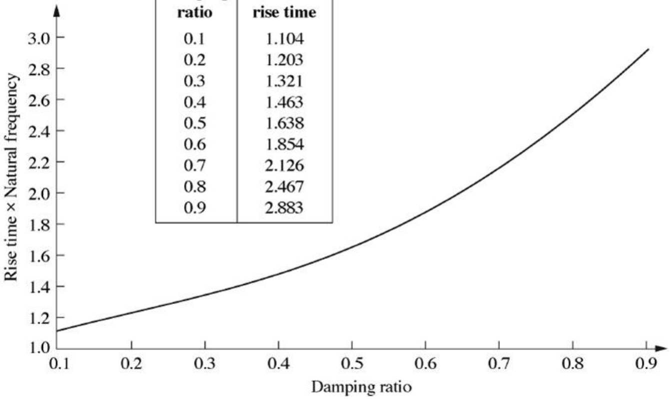
$$\omega t_1 = \cos^{-1}(.9) = .45, \qquad \omega_d t_2 = \cos^{-1}(.1) = 1.47$$

So that

$$T_r = t_2 - t_1 = \frac{1.02}{\omega_d} \qquad \textbf{WRONG!!!!}$$

However, with **Damping**, the situation changes.

- Damping *slows* the response.
- No easy formula for rise time of a complex pole!
- When $\zeta = .5$, $T_r \cong \frac{1.8}{\omega_n}$ (We will use this approximation)



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Recall the step response

$$y(t) = 1 - e^{\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

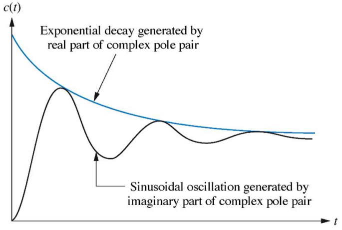
Oscillations are confined within an *Exponential Envelope*

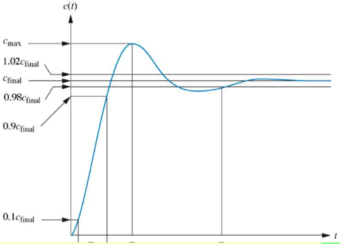
- The exponential envelop decays at rate σ.
- **Settling Time** for a complex pole is given by contraction of the envelope

$$\|1 - y(t)\| \leq e^{\sigma t} \leq .01$$

- Thus we use the same formula as for a real pole - i.e.

$$T_s = \frac{4.6}{-\sigma}$$



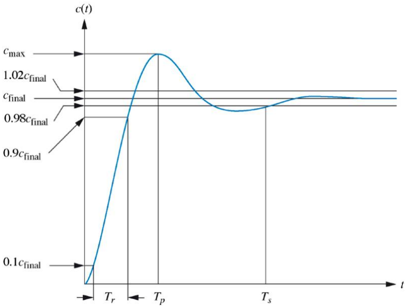


The time of MAXIMUM deflection.

- Only Complex poles have peaks.

Definition 1.

The **Peak Time**, T_p is time at which the step response obtains its maximum value.



To calculate T_p, we must find when

$$y(t) = 1 - e^{\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$
$$= 1 - e^{\sigma t}\frac{\omega_n}{\omega_d}\sin\left(\omega_d t + \phi\right)$$

Achieves its maximum.

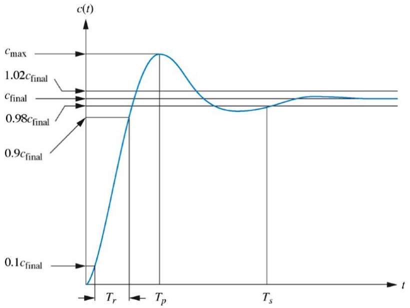
Complex Poles

How to Calculate Peak Time (T_p)

To find the extrema, we set $\dot{y}(t) = 0$, where it can be shown that

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{\sigma t} \sin \omega_d t$$
$$= \frac{\omega_n^2}{\omega_d} e^{\sigma t} \sin \omega_d t$$

So $\dot{y}(t) = 0$ when $t = \frac{n\pi}{\omega_d}$.



Because of the exponential envelope, the first peak will always be largest ($n = 1$).

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

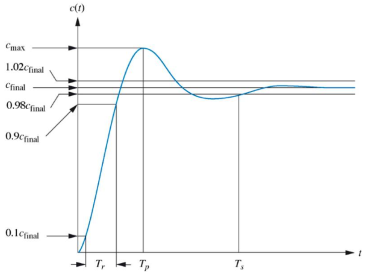
Unique to complex poles is the concept of overshoot:

Definition 2.

The **Percent Overshoot**, M_p is the peak value of the signal, as a percentage of steady-state ($\frac{c_{\max}-c_{final}}{c_{final}} \cdot 100\%$).

To find M_p, we need the max of $y(t)$ (We need $(c_{\max})$).

- $c_{\max} = y(T_p)$ where $T_p = \frac{\pi}{\omega_d}$.



Systems with high overshoot may move violently before settling.

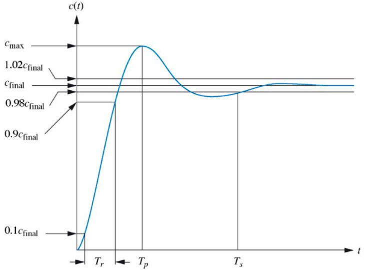
- May diverge from acceptable path
- Can cause crashes, un-modeled dynamics, etc.

To calculate M_p, we need the maximum value of $y(t)$.

- Occurs at time $T_p = \frac{\pi}{\omega_d}$.

Since $c_{final} = 1$ ($y_{ss} = 1$):

$$\begin{aligned} M_p &= y(T_p) - 1 \\ &= -e^{\sigma T_p}\left(\cos(\omega_d T_p) + \frac{\sigma}{\omega_d}\sin(\omega_d T_p)\right) \\ &= -e^{\sigma T_p}\left(\cos(\pi) + \frac{\sigma}{\omega_d}\sin(\pi)\right) \\ &= e^{\sigma T_p} = e^{\frac{\pi\sigma}{\omega_d}} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \end{aligned}$$



$$M_p = e^{\frac{\pi\sigma}{\omega_d}} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

M_p depends only on ζ.

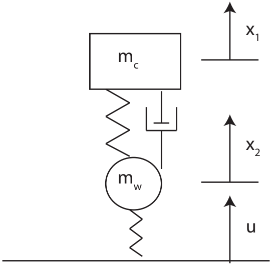
Estimate:

- Rise/Peak Time
- Percent Overshoot

Complex Poles

Numerical Example

Lets look at the suspension problem
Open Loop:



$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}$$

The poles are:

- $p_{1,2} = -.9567 \pm 1.2272\imath$
- $p_{3,4} = -.0433 \pm .6412\imath$

Because there are two sets of poles, we should consider both.

$$\sigma_1 = -.9567 \qquad \omega_{d,1} = 1.2272$$
$$\sigma_2 = -.0433 \qquad \omega_{d,2} = .6412$$

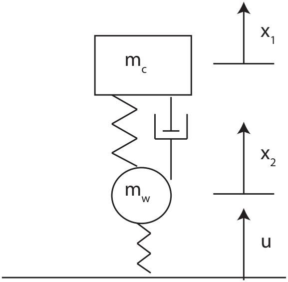
$$\omega_{n,1} = \sqrt{\sigma_1^2 + \omega_{d,1}^2} = 1.5561 \qquad \zeta_1 = \frac{|\sigma|}{\omega_{n,1}} = .6148$$
$$\omega_{n,2} = .6427 \qquad \zeta_2 = .0674$$

Complex Poles

Numerical Example

Closed Loop (Upper Feedback): Let $k = 1$

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + (3+k)s^2 + (1+k)s + (1+k)}$$



The poles are:

- $p_{1,2} = -.8624 \pm 1.4391\imath$
- $p_{3,4} = -.1376 \pm .8316\imath$

Consider both sets of poles.

$$\sigma_1 = -.8624 \qquad \omega_{d,1} = 1.4391$$
$$\sigma_2 = -.1376 \qquad \omega_{d,2} = .8316$$

The natural frequency and damping ratios are

$$\omega_{n,1} = 1.6777 \qquad \zeta_1 = .5140$$

$$\omega_{n,2} = .8429 \qquad \zeta_2 = .1632$$

Complex Poles

Numerical Example: Percent Overshoot

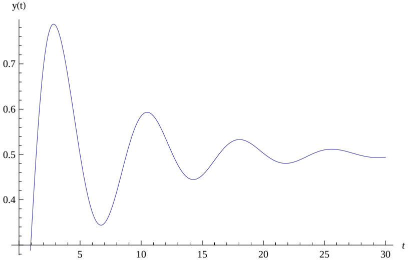


Figure: Closed Loop

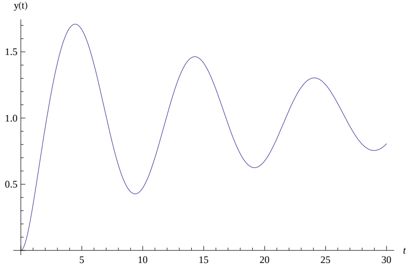


Figure: Open Loop

Overshoot: Open Loop (easiest to use σ and ω directly)

$$M_{p,1} = e^{\frac{\pi \sigma_1}{\omega_1}} = .0864 \qquad M_{p,2} = .8088$$

Overshoot: Closed Loop (Upper Feedback)

$$M_{p,1} = .152 \qquad M_{p,2} = .5946$$

A substantial improvement in performance.

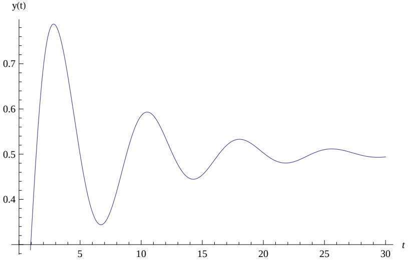


Figure: Closed Loop

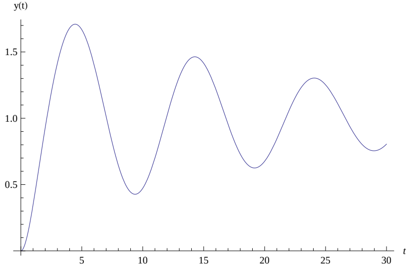


Figure: Open Loop

Settling Time: Open Loop

$$T_{s,1} = \frac{4.6}{-\sigma} = 4.81 \qquad T_{s,2} = 106.23$$

Settling Time: Closed Loop (Upper Feedback)

$$T_{s,1} = 5.33 \qquad T_{s,2} = 33.43$$

Complex Poles

Numerical Example: Peak Time

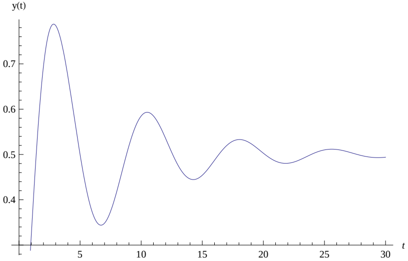


Figure: Closed Loop

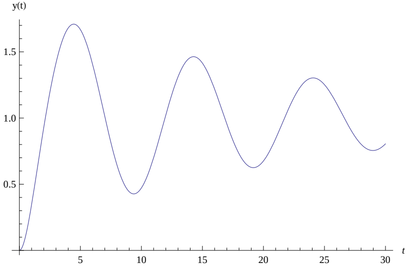


Figure: Open Loop

Peak Time: Open Loop

$$T_{p,1} = \frac{\pi}{\omega_d} = 2.56 \qquad T_{p,2} = 4.90$$

Peak Time: Closed Loop

$$T_{p,1} = 2.18 \qquad T_{p,2} = 3.78$$

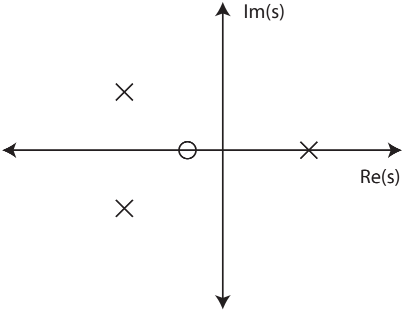
Performance Specifications

Dynamic response is determined by *pole locations*.

- Except Steady-State Error

Usually, dynamic response **improves** with feedback.

- Recall the numerical examples.
- Pole locations change under feedback.
- The choice of $k = 1$ was just a guess.



Consider: The goal of a controller is to change the location of the poles.

- But where do we want them?

Performance Specifications create Geometric Constraints in the Complex Plane.

Pole Locations

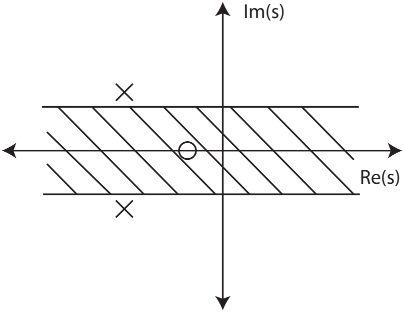
Constraint on Peak Time

Suppose we have performance specs for

- T_p, T_r, M_p etc.

We can translate this to regions of the complex plane.

Maximum Peak Time: $T_{p,desired}$.



We usually require $T_p < T_{p,desired}$.

$$\frac{\pi}{\omega_d} = T_p < T_{p,desired}$$

Which means

$$\omega_d > \frac{\pi}{T_{p,desired}}$$

The geometric interpretation is that the imaginary part be sufficiently large.

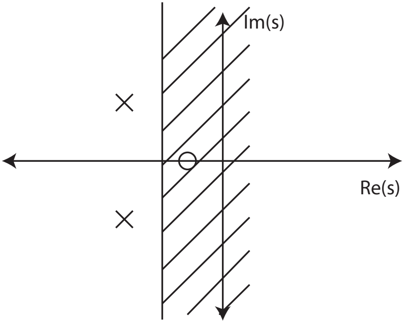
Maximum Settling Time: $T_{s,desired}$.

We want quick convergence.

- So we require $T_s < T_{s,desired}$.

Hence,

$$\frac{4.6}{-\sigma} = T_s < T_{s,desired}$$



Which translates to

$$\sigma < -\frac{4.6}{T_{s,desired}}$$

The geometric interpretation is that the real part be sufficiently negative.

Maximum Overshot: $M_{p,desired}$.

We don't like hitting things,

- So we need $M_p < M_{p,desired}$.

2 roots $(\sigma \pm \omega_d \imath)$ give two constraints:

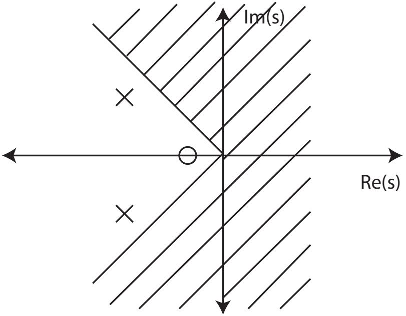
$$e^{\frac{\pi\sigma}{\pm\omega_d}} = M_p < M_{p,desired}$$

$$\frac{\pi\sigma}{\pm\omega_d} < \ln(M_{p,desired})$$

or since $\ln(M_{p,desired}) < 0$,

$$\omega_d < \frac{\pi}{\ln(M_{p,desired})}\sigma, \qquad \omega_d > -\frac{\pi}{\ln(M_{p,desired})}\sigma$$

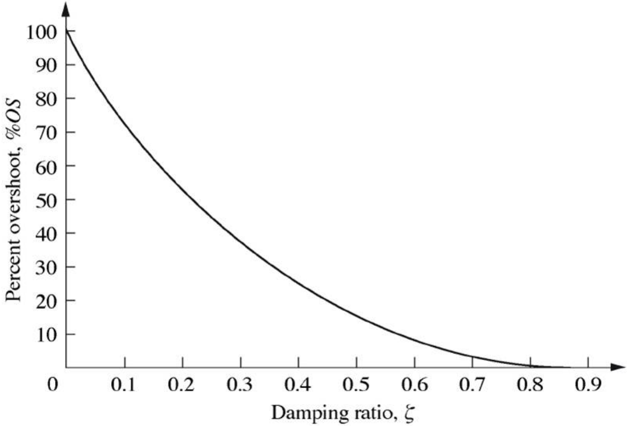
A sector constraint on σ and ω?



Complex Poles

Percent Overshoot

Alternatively, $M_{p,desired}$ is determined by damping ratio alone:



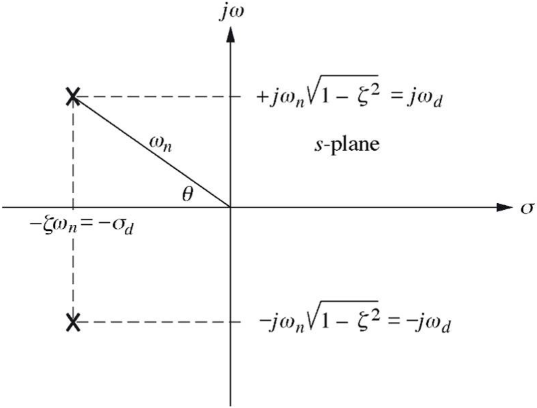
Invert:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}, \qquad \zeta = \frac{|\sigma|}{\omega_n}$$

Complex Poles

Percent Overshoot

A fixed $\zeta_{desired}$ defines an angle in the complex plane.



$$\theta = \frac{\pi}{2} - \sin^{-1}(\zeta_{desired})$$

Pole Locations

Constraint on Rise Time

The expression for rise time is complicated. We use $\zeta = .5$, to get

$$T_r \cong \frac{1.8}{\omega_n}$$

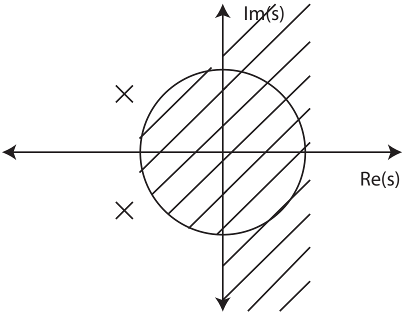
Maximum Rise Time: $T_{r,desired}$.

- We want quick response.
- We require $T_r < T_{r,desired}$.

$$\frac{1.8}{\omega_n} = T_r < T_{r,desired}$$

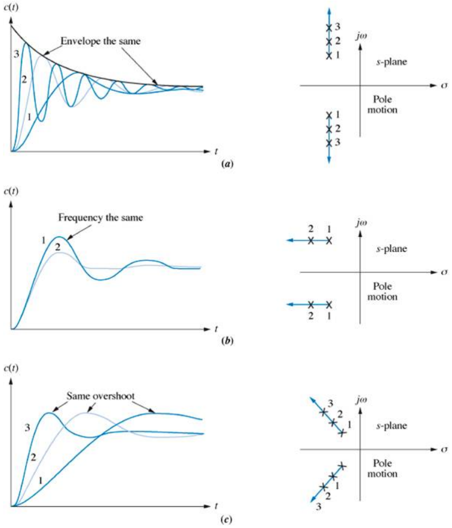
Thus we require

$$\omega_n > \frac{1.8}{T_{r,desired}}$$



Recall that $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$, so the geometric interpretation is a circle: $\|s\| > \frac{1.8}{T_{r,desired}}$.

Complex Poles



Pole Locations

Multiple Constraints

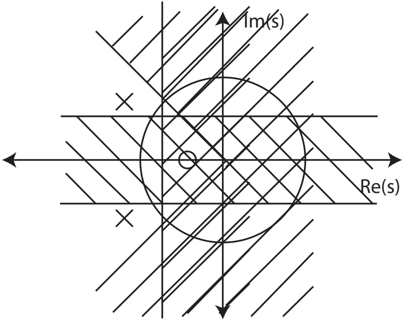
Mostly, we have several constraints

$$\omega_d > \frac{\pi}{T_{p,desired}}$$

$$\sigma < -\frac{4.6}{T_{s,desired}}$$

$$\omega_d < \frac{\pi}{\ln(M_{p,desired})}\sigma$$

$$\omega_n > \frac{1.8}{T_{r,desired}}$$



Any pole locations not prohibited are allowed.

High Performance Aircraft:

- **Overshoot:** Reduce overshoot to less than 5%.

$$M_{p,desired} = .05$$
$$\omega_d < \frac{\pi}{\ln(M_{p,desired})}\sigma = -1.05\sigma$$

- A difficult requirement to meet?
- **Rise Time:** Quick response is critical. Limit Rise Time to 1s or less

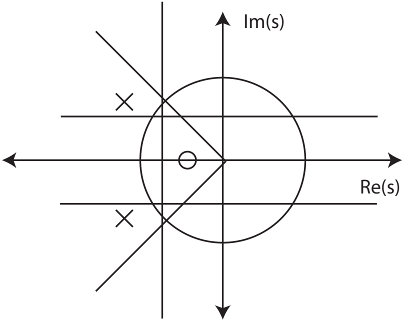
$$T_{r,desired} = 1$$
$$\omega_n > \frac{1.8}{T_{r,desired}} = 1.8$$

- **Settling Time:** Limit settling time to $T_{s,desired} = 3.5s$.

$$T_{s,desired} = 3.5s$$
$$\sigma < -\frac{4.6}{T_{s,desired}} = -1.333$$

We have the required
Overshoot: Along a line of about

$$\theta = \text{atan}\left(\frac{\omega_d}{\sigma}\right) = \text{atan}\left(\frac{1}{-.9535}\right) = 46^\circ$$



Which means a damping ratio of
$\zeta = \sin(90 - 46^\circ) = .69$.

- Roughly $\omega_d = \sigma$

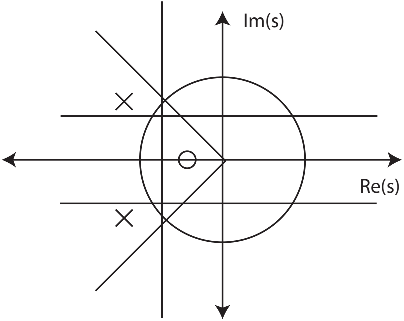
To satisfy T_s, $\sigma < -1.333$, so lets try $\sigma = -1.5$.

- Then $\omega_d < 1.5$
- Choose $\omega_d = 1.4$

For T_r, need $\omega_n > 1.8$. However,

$$\omega_n = \sqrt{\omega_d^2 + \sigma^2} = 2.05$$

So rise time is already satisfied.



If we need to decrease rise time, increase omega, while staying on lines of constant overshoot

Pole Locations

Missile Video

Estimate Performance Specs:

Summary

What have we learned today?

In this Lecture, you will learn:

Characteristics of the Response

Complex Poles

- Rise Time
- Settling Time
- Percent Overshoot

Performance Specifications

- Geometric Pole Restrictions

Next Lecture: Designing Controllers