Systems Analysis and Control

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Lecture 11: Proportional, Integral and Differential Control

Overview

In this Lecture, you will learn:

Limits of Proportional Feedback

- Performance Specifications.

Derivative Feedback

- Pros and Cons
- PD Control
- Pole Placement

More on Steady-State Error

- Response to ramps and parabolae
- Limits of PD control

Integral Feedback

- Elimination of steady-state error
- Pole-Placement

Recall the Inverted Pendulum Problem

Proportional Feedback cannot meet any performance specs

Transfer Function

$$\hat{G}(s) = \frac{1}{Js^2 - \frac{Mgl}{2}}$$

For a simple proportional gain: $\hat{K}(s) = k$

Closed Loop Transfer Function (Lower Feedback Interconnection):

$$\frac{GK}{1+GK} = \frac{k}{Js^2 - \frac{Mgl}{2} + k}$$

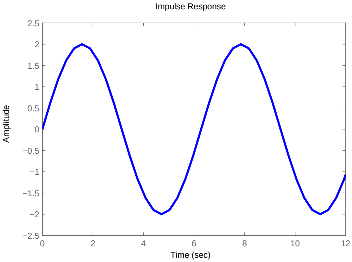


Figure: Case 1: $k > \frac{Mgl}{2}$

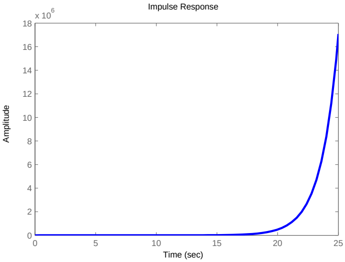


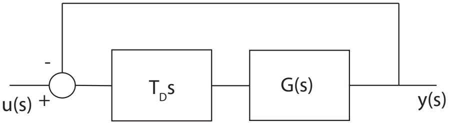
Figure: Case 2: $k < \frac{Mgl}{2}$

Both cases are unstable!

Differential Control

Now suppose we furthermore have a performance specification:

- **Overshoot**
- **Rise Time**
- **Settling Time**



Problem: There is no solution using proportional gain: $\hat{K}(s) = k$.

Now we must consider a New Kind of Controller:
Derivative Control: Choose $\hat{K}(s) = T_D s$

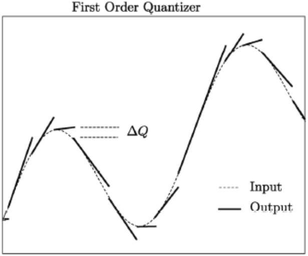
The controller is of the form

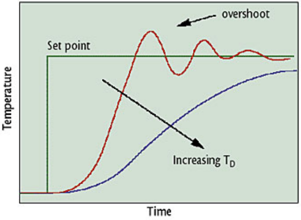
$$u(t) = T_D \dot{e}(t)$$

The controller is called **Differential/Derivative Control** because it is proportional to the rate of change of the error.

Differential Control (Predicting the Future)

Differential control improves performance by reacting quickly.





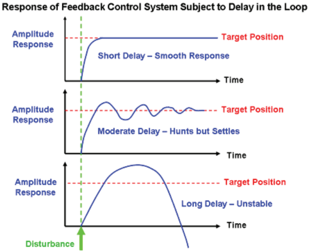
Prediction:

- To measure $\dot{y}(t)$, recall the definition of derivative:

$$\dot{y}(t) \cong \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

- The $\dot{y}(t)$ term depends on both the current position and predicted position.
  - A way to speed up the response (or slow it down).

Differential Control: Implemented using Delay (Dangerous!)



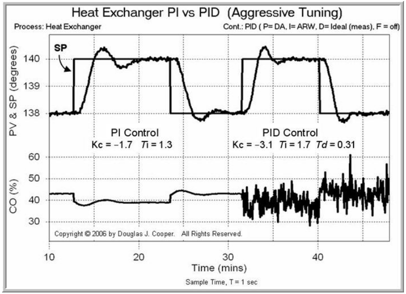
Problem: Differential control is implemented using delay.

- $y(t)$ is the measurement.
- $\dot{y}(t)$ cannot be measured directly
  - Approximate using the delayed response:

$$\dot{y}(t) \cong \frac{y(t) - y(t - \Delta t)}{\Delta t}$$

  - Delay can cause instabilities.

Differential Control: Produces Noise (Dangerous!)



Noise Amplification:

- Measurement of $\dot{y}(t)$ is heavily influenced by noise.

$$\dot{y}(t) \cong \frac{y(t) - y(t - \Delta t)}{\Delta t}$$

- Sensor measurements have error ($\tilde{y} = y \pm \sigma$)
- As $\Delta t \to 0$, the effect of noise, σ is amplified:

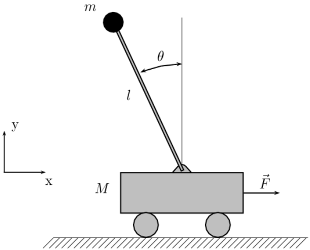
$$\dot{\tilde{y}}(t) = \frac{y(t) - y(t - \Delta t)}{\Delta t} + \frac{2\sigma}{\Delta t} \to \infty$$

Derivative Control Alone Rarely Works

Useless for Inverted Pendulum

Controller: $\hat{K}(s) = T_D s$
Closed Loop Transfer Function:

$$\frac{T_D/Js}{s^2 + T_D/Js - \frac{Mgl}{2J}}$$



2nd-Order System As we learned last lecture, stable iff *both*

- $T_D/J > 0$
- $-\frac{Mgl}{2J} > 0$

Derivative Feedback **Alone** cannot stabilize a system.

Proportional-Derivative (PD) Control

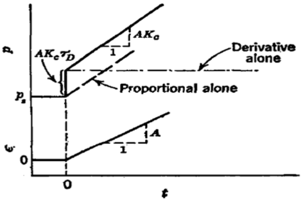


Figure: Proportional and Derivative Response to Ramp input

Differential Control is usually combined with proportional control.

- To improve stability
- To reduce steady-state error.
- To reduce the effect of noise.

Controller: The form of control is

$$u(t) = K\left[e(t) + T_D\dot{e}(t)\right]$$

or

$$\hat{u}(s) = K\left[1 + T_D s\right]\hat{e}(s)$$

PD Control - Effect on CL Transfer Function

Applied to a 2nd-order system

Lets look at the effect of PD control on a 2nd-order system:

$$\hat{G}(s) = \frac{1}{s^2 + bs + c}$$

Controller: $\hat{K}(s) = K\left[1 + T_D s\right]$
Closed Loop Transfer Function:

$$\frac{\hat{K}(s)\hat{G}(s)}{1 + \hat{K}(s)\hat{G}(s)} = \frac{K\left[1 + T_D s\right]}{s^2 + bs + c + K\left[1 + T_D s\right]}$$
$$= \frac{K\left[1 + T_D s\right]}{s^2 + (b + KT_D)s + (c + K)}$$

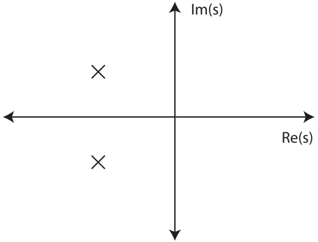
The poles of the system are freely assignable for a 2nd order system.

- The *Gains* T_D and K allow us to construct any denominator we desire.

Generic PD Control - Effect on Pole Locations

Applied to a 2nd-order system

Suppose we want poles at $s = p_1, p_2$.



- We want the closed loop of the form:

$$\frac{1}{(s-p_1)(s-p_2)} = \frac{1}{(s^2-(p_1+p_2)s+p_1p_2)}$$

Thus we want

- $c + K = p_1 p_2$ which means $K = p_1 p_2 - c$.
- $b + KT_D = -(p_1 + p_2)$ which means $T_D = -\frac{p_1+p_2+b}{K} = -\frac{p_1+p_2+b}{p_1p_2-c}$

PD feedback gives Total Control over a 2nd-order system.

Generic PD Control Example

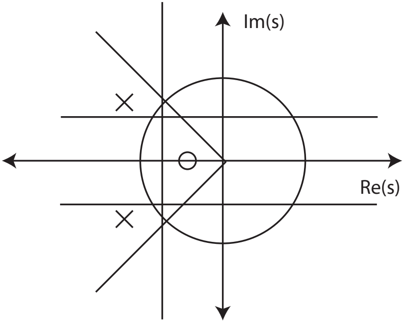
Pole Placement: Meet Performance Specs

Suppose we have the 2nd-order system

$$\hat{G}(s) = \frac{1}{s^2 + s + 1}$$

and performance specifications:

- **Overshoot:** $M_{p,desired} = .05$
- **Rise Time:** $T_{r,desired} = 1s$
- **Settling Time:** $T_{s,desired} = 3.5s$.



As we found in Lecture 9, these specifications mean that the poles satisfy:

$$\sigma < -.9535\omega, \qquad \sigma < -1.333, \qquad \omega_n > 1.8$$

We chose the pole locations:

$$s = -1.5 \pm 1.4\imath$$

Generic PD Control Example

Pole Placement: Determine gains K and T_D

The desired system is

$$\frac{1}{(s^2 - (p_1 + p_2)s + p_1 p_2)}$$

The closed loop is

$$\frac{K\left[1 + T_D s\right]}{s^2 + (b + KT_D)s + (c + K)}$$

To get the pole locations:

$$p_{1,2} = -1.5 \pm 1.4\imath$$

we choose

- The Proportional Gain (K):

$$K = p_1 p_2 - c = (-1.5 + 1.4\imath)(-1.5 - 1.4\imath) + 1 = 1.5^2 + 1.4^2 - 1 = 3.21$$

- The Derivative Gain (T_D)

$$T_D = -\frac{p_1 + p_2 + b}{K} = -\frac{-3 + 1}{3.21} = \frac{2}{3.21} = .623$$

This gives the controller:

$$\hat{K}(s) = K(1 + T_D s) = 3.21 + 2s$$

PD Control has NO effect on Steady-State Error

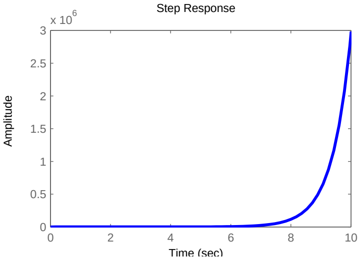


Figure: Open Loop

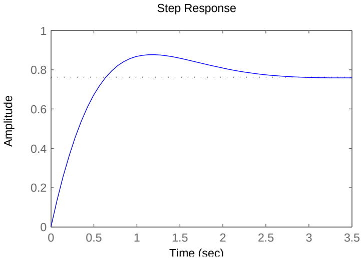


Figure: Closed Loop

Although the PD controller gives us control of the pole locations, the steady-state value is

$$y_{ss} = \frac{K}{c+K} = \frac{3.21}{4.21} = .7625$$

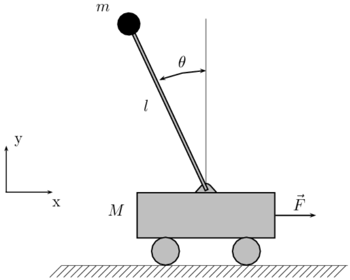
PD Control Example

Inverted Pendulum

Lets look at the effect of PD control on the inverted Pendulum:

$$\hat{G}(s) = \frac{1/J}{s^2 - \frac{Mgl}{2J}}$$

Controller: $K\left[1 + T_D s\right]$



Closed Loop Transfer Function:

$$\frac{\hat{K}(s)\hat{G}(s)}{1+\hat{K}(s)\hat{G}(s)} = \frac{K/J\left[1 + T_D s\right]}{s^2 - \frac{Mgl}{2J} + K/J\left[1 + T_D s\right]}$$

$$= \frac{K/J\left[1 + T_D s\right]}{s^2 + K/JT_D s + (K/J - \frac{Mgl}{2J})}$$

PD Control Example

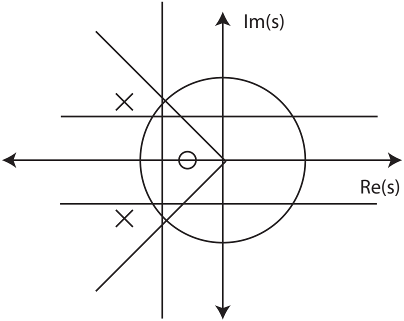
Inverted Pendulum : Desired Pole Locations

To achieve the performance specifications:

- **Overshoot:** $M_{p,desired} = .05$
- **Rise Time:** $T_{r,desired} = 1s$
- **Settling Time:** $T_{s,desired} = 3.5s$.

We want poles at

$$s = -1.5 \pm 1.4\imath$$



Thus we want

- $c + K = p_1 p_2$ which means $K = p_1 p_2 - c$.
- $b + KT_D = -(p_1 + p_2)$ which means

$$T_D = -\frac{p_1 + p_2 + b}{K} = -\frac{p_1 + p_2 + b}{p_1 p_2 - c}$$

PD Control Example

Inverted Pendulum

The closed loop is

$$\frac{K/J\,[1+T_D s]}{s^2+K/JT_D s+(K/J-\frac{Mgl}{2J})}$$

To get the pole locations $p_{1,2}=-1.5\pm 1.4\imath$
we choose

- The Proportional Gain (K):

$$K/J=p_1p_2-c=4.21+\frac{Mgl}{2J}$$

- The Derivative Gain (T_D):

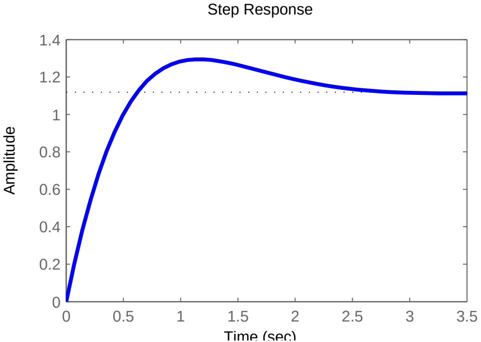
$$T_D=-\frac{p_1+p_2+b}{p_1p_2-c}=\frac{3}{4.21+\frac{Mgl}{2J}}$$

This gives the controller:

$$\hat{K}(s)=K(1+T_D s)=4.21J+\frac{1}{2}Mgl\left(1+\frac{3}{4.21+\frac{Mgl}{2J}}s\right)$$

PD Control Example

Inverted Pendulum: No Effect on Steady-State Error



The steady-state error with this controller is $(K = J = M = g = l = 1)$

$$y_{ss} = \frac{K/J}{(K/J - \frac{Mgl}{2J})} = \frac{4.21}{4.21 - .5} = 1.135$$

Derivative Control has **No Effect** on the steady-state error!

Lets take another look at steady-state error

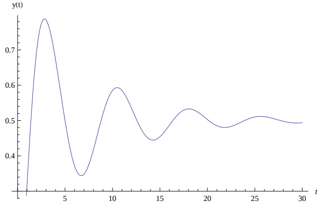


Figure: Suspension Response for $k = 1$

Problems:

- If target is moving, we may never catch up.
- Even if we can catch a moving target, we may not catch an accelerating target.

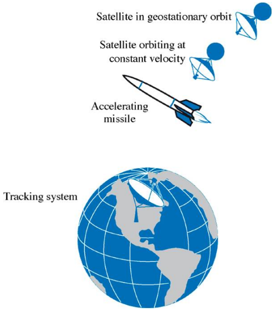
For these problems, the step response is not appropriate.

Recall:

- We measured steady-state error using the step response.
  - $e_{ss} = 1 - \lim_{t\to\infty} y(t)$

Sometimes this doesn't work.

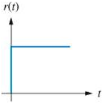
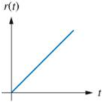
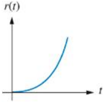
- Assumes objective doesn't move.



Ramp and Parabolic Inputs

There are other types of response we can consider.

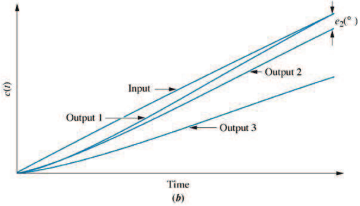
TABLE 7.1 **Test waveforms for evaluating steady-state errors of position control systems**

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

- Ramp response tracks error for a target with constant velocity.
- Parabolic response tracks error for a target with a constant acceleration.

Ramp and Parabolic Inputs

We can use the final value theorem to find the response to ramp and parabolic inputs:



Ramp Response:

Recall the ramp input:

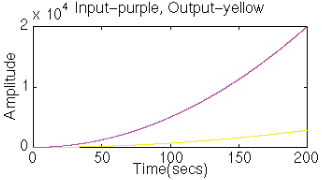
$$u(t) = t \qquad \hat{u}(s) = \frac{1}{s^2}$$

The steady-state error of $\hat{G}$ to a ramp input is

$$e_{ss} = \lim_{s\to 0} s\hat{e}(s) = \lim_{s\to 0} s(1-\hat{G}(s))\hat{u}(s) = \lim_{s\to 0} \frac{1-\hat{G}(s)}{s}$$

Ramp and Parabolic Inputs

We can use the final value theorem to find the response to parabolic inputs:



Parabolic Response:
Recall the parabolic input:

$$u(t) = t^2 \qquad \hat{u}(s) = \frac{1}{s^3}$$

The steady-state error in response of $\hat{G}$ to a parabolic input is

$$e_{ss} = \lim_{s \to 0} s(\hat{u}(s) - \hat{y}(s)) = s(1 - \hat{G}(s))\hat{u}(s) = \frac{1 - \hat{G}(s)}{s^2}$$

Note: The steady-state error to a parabolic input is usually infinite.

Ramp and Parabolic Inputs

The effect of the numerator

For steady-state error, the numerator of the transfer function becomes important: for

$$\hat{G}(s) = \frac{n(s)}{d(s)}$$

Steady state error of $\hat{G}$ is

$$e_{ss} = \lim_{s\to 0}(1-\hat{G}(s))s\hat{u}(s) = \lim_{s\to 0}\left(\frac{d(s)}{d(s)} - \frac{n(s)}{d(s)}\right)s\hat{u}(s)$$
$$= \lim_{s\to 0}\frac{d(s)-n(s)}{d(s)}s\hat{u}(s)$$

$\hat{u}(s)$ is the test signal

- **Step Input:** $s\hat{u}(s) = 1$
- **Ramp Input:** $s\hat{u}(s) = \frac{1}{s}$
- **Parabolic Input:** $s\hat{u}(s) = \frac{1}{s^2}$

Error Signals for Systems in Feedback

Use $\hat{G}(s) = \frac{n(s)}{d(s)}$

Lower Feedback Interconnection: $\dfrac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)} = \dfrac{n(s)\hat{K}(s)}{d(s)+n(s)\hat{K}(s)}$

SS error for Lower Feedback Interconnection:

$$\hat{e}(s) = \left(1 - \frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)}\right) s\hat{u}(s) = \left(\frac{1}{1+\hat{G}(s)\hat{K}(s)}\right) s\hat{u}(s)$$

Step Response:

$$e_{ss,step} = \lim_{s\to 0} \frac{1}{1+\hat{G}(s)\hat{K}(s)} = \lim_{s\to 0} \frac{d(s)}{d(s)+n(s)\hat{K}(s)}$$

Ramp Response:

$$e_{ss,ramp} = \lim_{s\to 0} \frac{1}{1+\hat{G}(s)\hat{K}(s)} \frac{1}{s} = \lim_{s\to 0} \frac{d(s)}{d(s)+n(s)\hat{K}(s)} \frac{1}{s}$$

Parabolic Response:

$$e_{ss,parabola} = \lim_{s\to 0} \frac{1}{1+\hat{G}(s)\hat{K}(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{d(s)}{d(s)+n(s)\hat{K}(s)} \frac{1}{s^2}$$

Proportional Control Can Make Ramp Response Worse!!!

Consider the Suspension Example: **Open Loop:**

$$\hat{G}(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+s+1}$$

$$1-\hat{G}(s) = \frac{s^4+2s^3+3s^2+s+1-s^2-s-1}{s^4+2s^3+3s^2+s+1} = \frac{s^4+2s^3+2s^2}{s^4+2s^3+3s^2+s+1}$$

Ramp Response:

$$\lim_{s\to 0} \frac{1-\hat{G}(s)}{s} = \lim_{s\to 0} \frac{s^3+2s^2+2s}{s^4+2s^3+3s^2+s+1} = 0$$

What happens when we close the loop?
Closed Loop Transfer Function:

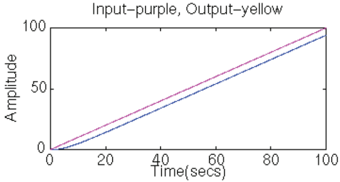
$$\frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}$$

Ramp Response:

$$e_{ss,ramp} = \lim_{s\to 0} \frac{1}{s(1+\hat{G}(s)\hat{K}(s))} \cong \lim_{s\to 0} \frac{s^4+2s^3+3s^2+s+1}{k(s^2+s+1)}\frac{1}{s} = \infty$$

Proportional response isn't capable of controlling a ramp input

Example of Ramp Response



The only way to control a ramp input using feedback is to put a pole at the origin:

Controller: $\hat{K}(s) = \frac{1}{T_I s}$

Ramp Response:

$$e_{ss,ramp} = \lim_{s \to 0} \frac{d(s)}{d(s) + n(s)\hat{K}(s)} \frac{1}{s} = \lim_{s \to 0} \frac{d(s)}{sd(s)T_I + n(s)} \frac{T_I s}{s} = \frac{d(0)}{n(0)} T_I$$

By including $1/s$ in the controller, the steady-state error becomes finite.

Integral Control is Used to Eliminate Steady-State Error

The purpose of integral control is primarily to eliminate steady-state error.
Controller: The form of control is

$$u(t) = \frac{1}{T_I} \int_0^t e(\theta) d\theta$$

or, in the Laplace transform

$$\hat{u}(s) = \frac{1}{T_I s} \hat{e}(s)$$

One must be careful when using integral feedback

- By itself, an integrator is unstable.
  - A pole at the origin.

Integral Control is Often Destabilizing

Suspension Problem Again

Now lets re-examine the suspension problem

Integral Control Alone: $\hat{K}(s) = \frac{1}{T_I s}$

Closed Loop Transfer Function (Lower Feedback):

$$\frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)} = \frac{s^2+s+1}{T_I s^5 + 2T_I s^4 + 3T_I s^3 + (T_I+1)s^2 + (T_I+1)s + 1}$$

If we set $T_I = .1$, then the transfer function has poles at

- $p_{1,2} = -.55 \pm .89\imath$, $p_3 = -2.26$, $p_{4,5} = .6384 \pm 1.877\imath$

Integral feedback can Destabilize the system where proportional feedback couldn't!

Integral Control is Always Combined with Proportional Control

And Sometimes with Differential Control

Integral Feedback Alone is destabilizing!

PI Feedback: Proportional-Integral

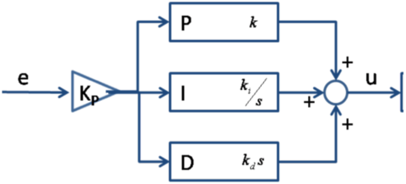
$$u(t) = K\left(e(t) + \frac{1}{T_I}\int_0^t e(\theta)d\theta\right)$$

$$\hat{K}(s) = K\left(1 + \frac{1}{T_I s}\right)$$

PID Feedback:
Proportional-Integral-Differential

$$u(t) = K\left(e(t) + \frac{1}{T_I}\int_0^t e(\theta)d\theta + T_D\dot{e}(t)\right)$$

$$\hat{K}(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$$



PID Control

Example

Finally, lets see the effect of PID control on a second-order system:

$$\hat{G}(s) = \frac{1}{s^2 + bs + c} \qquad \hat{K}(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$$

Closed Loop:

$$\begin{aligned}
\frac{\hat{G}\hat{K}}{1+\hat{G}\hat{K}} &= \frac{K\left(1 + \frac{1}{T_I s} + T_D s\right)}{s^2 + bs + c + K\left(1 + \frac{1}{T_I s} + T_D s\right)} \\
&= \frac{K\left(s + \frac{1}{T_I} + T_D s^2\right)}{s^3 + bs^2 + cs + K\left(s + \frac{1}{T_I} + T_D s^2\right)} \\
&= \frac{K T_D s^2 + Ks + K\frac{1}{T_I}}{s^3 + (b + KT_D)s^2 + (c + K)s + \frac{K}{T_I}}
\end{aligned}$$

Steady-State Response:

$$y_{ss,step} = \frac{\frac{K}{T_I}}{\frac{K}{T_I}} = 1$$

No Steady-State Error!

PID Control Can Be Used For Pole Placement

For a Second-Order System

Pole Placement: The three pole locations can be determined exactly.

- Given three poles: p_1, p_2, p_3.
- Construct Desired denominator:

$$\frac{1}{(s-p_1)(s-p_2)(s-p_3)} = \frac{1}{s^3 + a_d s^2 + b_d s + c_d}$$

Three equations:

- $b + KT_D = a_d$
- $c + K = b_d$
- $\frac{K}{T_I} = c_d$

Which can be solved as

- $K = b_d - c$
- $T_I = \frac{K}{c_d}$
- $T_D = \frac{a_d - b}{K}$

Summary

What have we learned today? In this Lecture, you learned:

Limits of Proportional Feedback

- Performance Specifications.

Derivative Feedback

- Pros and Cons
- PD Control
- Pole Placement

More on Steady-State Error

- Response to ramps and parabolae
- Limits of PD control

Integral Feedback

- Elimination of steady-state error
- Pole-Placement