Systems Analysis and Control

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Lecture 12: Root Locus

Overview

In this Lecture, you will learn:

Review of Feedback

- Closing the Loop
- Pole Locations

Changing the Gain

- Numerical Examples
  - Pole Locations
- Routh-Hurwitz vs. Root Locus

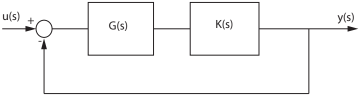
A Review of Complex Numbers

- Polar Form
- Multiplication-Division

The Effect of Feedback

Feedback changes the open loop

$$\hat{G}(s) = \frac{n_G(s)}{d_G(s)} \qquad \hat{K}(s) = \frac{n_K(s)}{d_K(s)}$$



to

$$\frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)} = \frac{n_G(s)n_K(s)}{d_G(s)d_K(s)+n_G(s)n_K(s)}$$

The pole locations are the roots of

$$d_G(s)d_K(s)+n_G(s)n_K(s) = 0$$

Objective: A closed loop denominator.

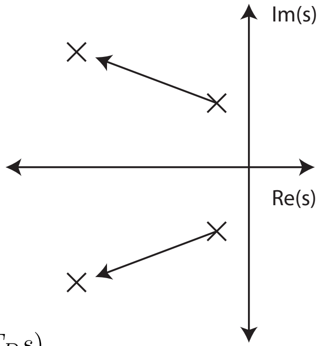
$$d(s) = (s-p_1)\cdots(s-p_n)$$

Big Question: How to choose $n_K(s)$ and $d_K(s)$ so that

$$d(s) = d_G(s)d_K(s)+n_G(s)n_K(s)$$

For a **Second-Order System**

$$\hat{G}(s) = \frac{1}{s^2 + as + b}$$



with PD feedback

$$\hat{K}(s) = K\left(1 + T_D s\right)$$

We can achieve any denominator

$$d(s) = s^2 + cs + d$$

Question: What happens for more complicated systems:

Open Loop:

$$\frac{s^2+s+1}{s^4+2s^3+3s^2+s+1}$$

Closed Loop:

$$\frac{K\left(1+T_Ds\right)\left(s^2+s+1\right)}{s^4+2s^3+3s^2+s+1+K\left(1+T_Ds\right)\left(s^2+s+1\right)}$$

$$=\frac{K\left(T_Ds^3+(1+T_D)s^2+(1+T_D)s+1\right)}{s^4+(2+KT_D)s^3+(3+K+KT_D)s^2+(1+K+KT_D)s+1+K}$$

Given a desired denominator

$$d(s)=s^4+as^3+bs^2+cs+d$$

Which gives 4 equations and 2 unknowns

$$a=2+KT_D \qquad b=3+K+KT_D$$

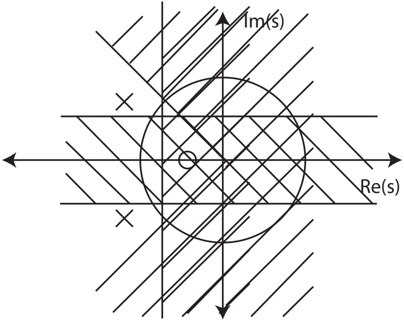
$$c=1+K+KT_D \qquad d=1+K$$

There is No Solution!!!.

We rarely need to achieve a precise set of poles.

Performance Specifications Determine **Regions** of the Complex Plane.

- Stability
- Rise Time
- Settling time
- Overshoot



New Question: What controller will ensure all roots of

$$d_G(s)d_K(s) + n_G(s)n_K(s)$$

lie in the desired region of the complex plane.

More fundamentally, how does changing $n_K(s)$ and $d_K(s)$ change the roots of

$$d_G(s)d_K(s) + n_G(s)n_K(s)?$$

The answer is complicated

- Must account for the effect of each term in n_K and d_K

So simplify, lets consider a controller with only a single free parameter.

$$\hat{K}(s) = k$$

Other options include:

- **PD Control**: $\hat{K}(s) = 1 + T_D s$
- **PI Control**: $\hat{K}(s) = 1 + \frac{1}{T_I s}$

Question: How do the roots of

$$d_G(s) + k n_G(s)?$$

change with k?

$$\hat{G}(s) = \frac{n_G(s)}{d_G(s)}$$

Definition 1.

The **Root Locus** of $\hat{G}(s)$ is the set of all poles of

$$\frac{k\hat{G}(s)}{1 + k\hat{G}(s)} = \frac{n_G(s)}{d_G(s) + kn_G(s)}$$

as k ranges from 0 to ∞

Alternatively:

- The roots of $1 + k\hat{G}(s)$ for $k \geq 0$
- The roots of $d_G(s) + kn_G(s)$ for $k > 0$
- The solutions of $\hat{G}(s) = \frac{-1}{k}$ for $k \geq 0$
- Values of s where $\angle G(s) = 180°$ (Woah... Not so fast)

We can estimate the root locus by finding the roots for several different values of k

Example: Video Surveillance System.

Pole at $s = 0$ to eliminate steady-state error.

Open Loop:

$$\hat{G}(s) = \frac{1}{s(s+10)}$$

Closed Loop:

$$\frac{k}{s^2 + 10s + k}$$

Pole Locations:

$$p_{1,2} = -5 \pm \frac{1}{2}\sqrt{100 - 4k}$$

Root Locus

Video Surveillance System

TABLE 8.1 **Pole location as function of gain for the system of Figure 8.4**

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	−5 + *j*2.24	−5 − *j*2.24
35	−5 + *j*3.16	−5 − *j*3.16
40	−5 + *j*3.87	−5 − *j*3.87
45	−5 + j4.47	−5 − *j*4.47
50	−5 + *j*5	−5 − *j*5

$$p_{1,2} = -5 \pm \frac{1}{2}\sqrt{100 - 4k}$$

Root Locus

Video Surveillance System

We can visualize the effect of changing k by plotting the poles on the complex plane.

TABLE 8.1 **Pole location as function of gain for the system of Figure 8.4**

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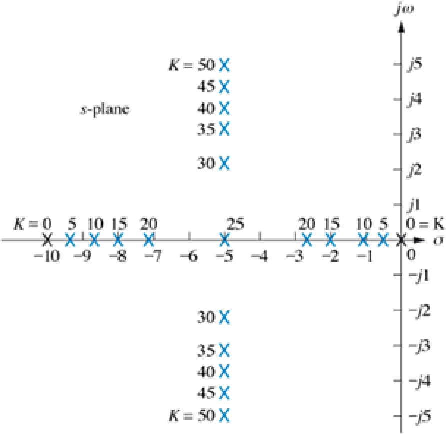
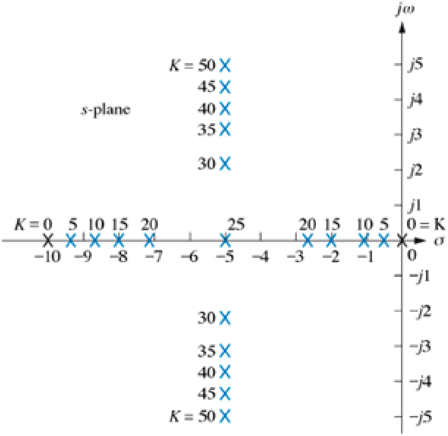
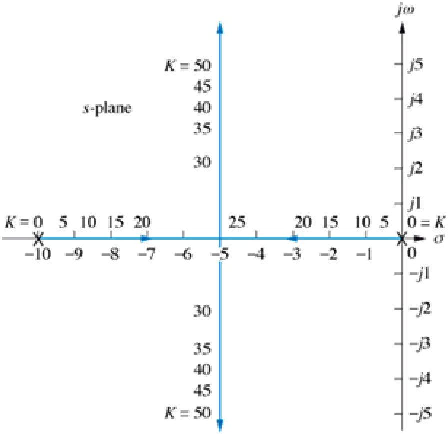
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



Plotting every possible value of k yields the *root locus*.

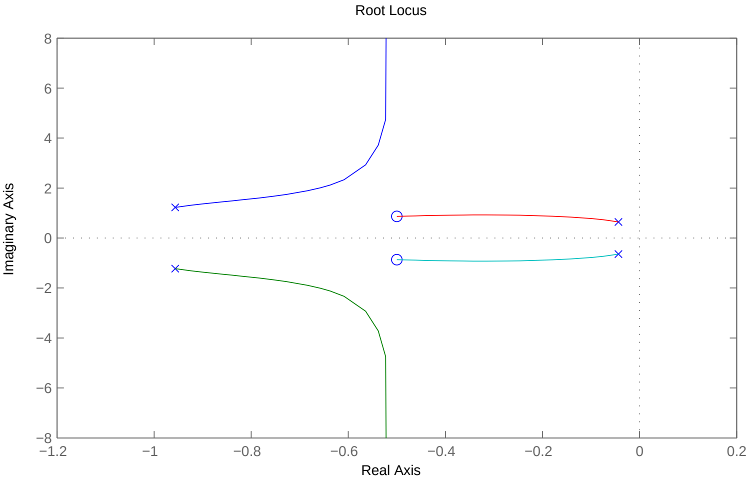




Connect the dots.

Root Locus

Example: Suspension System

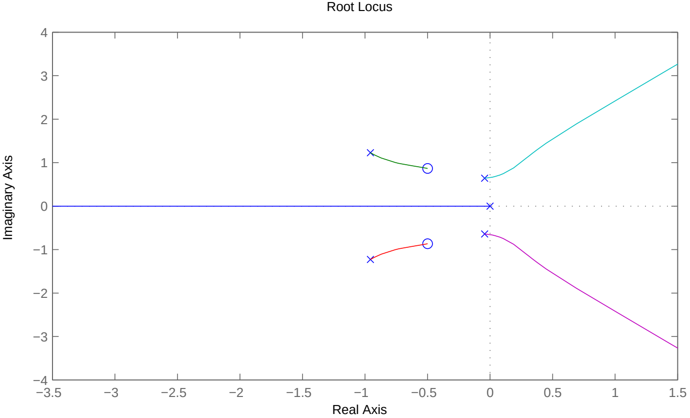


From Routh Test: Stable for all $k > 0$.

Root Locus

Example: Suspension System with Integral Feedback

Now, if we add integral feedback: $\hat{K}(s) = k\frac{1}{s}$

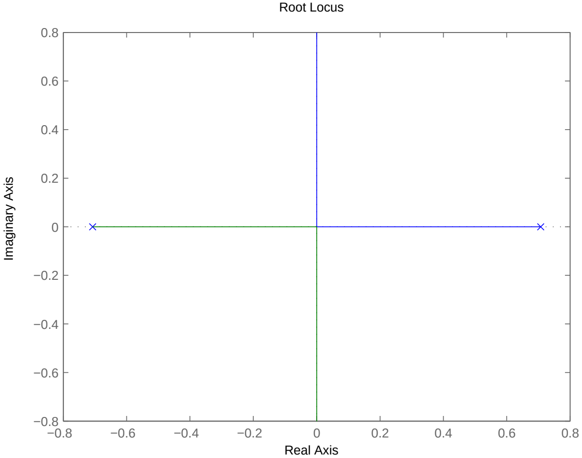


From Routh Test: Stable for all $k < .1$.

Root Locus

Example: Inverted Pendulum Model

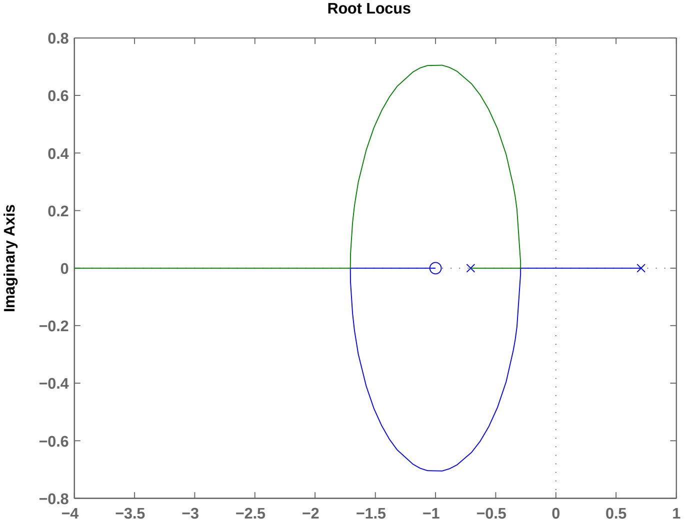
$$\hat{G}(s) = \frac{1}{s^2 - \frac{1}{2}}$$



Root Locus

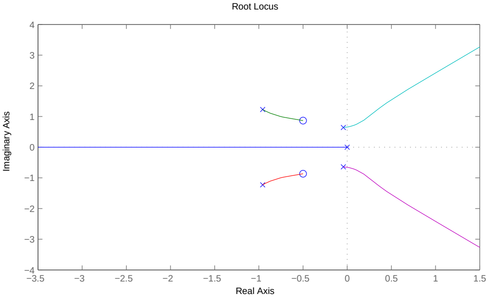
Example: Inverted Pendulum Model

Now an inverted pendulum with some derivative feedback: $\hat{K}(s) = k(1+s)$



Root Locus

Complex Numbers



When Matlab calculates the root locus, it plots every point.

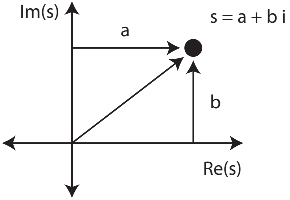
- Impractical for students
- Yields no intuition.
  - Root Locus is only one parameter.
  - We must know how to manipulate the root locus by changes in controller type.

Before we analyze the root locus, we begin with a review of Complex Numbers.

Consider a Complex Number:

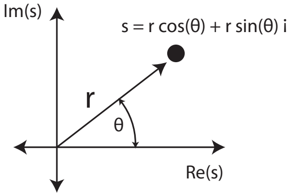
$$s = a + b\imath$$

The *Complex Plane* is the a-b plane.



A complex number can also be represented in polar form

$$s = r\left(\cos\theta + \imath \sin\theta\right)$$



Recall the Euler equation

$$e^{\imath\theta} = \cos\theta + \imath \sin\theta$$

Euler yields the more practical form:

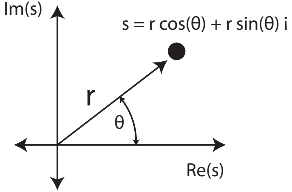
$$s = re^{\theta\imath}$$

Rectilinear

$$s = a + b\imath$$

Polar

$$s = re^{\theta\imath}$$



Notation:

- r is called the **Magnitude**
  - Denoted $r = |s|$
- $\theta =$ is called the **Phase**
  - Denoted $\theta = \angle s$

The relationship between Polar and Rectilinear coordinates is obvious

- $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
- $r = \sqrt{a^2 + b^2}$

- $a = r\cos\theta$
- $b = r\sin\theta$

Complex Numbers

Multiplication

In polar form, Multiplying and Dividing complex numbers is cleaner.

$$s_1 = r_1 e^{\theta_1 \imath} \qquad s_1 = a_1 + b_1 \imath$$

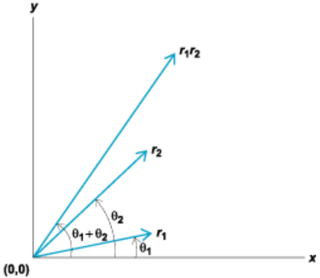
$$s_2 = r_2 e^{\theta_2 \imath} \qquad s_2 = a_2 + b_2 \imath$$

$$\begin{aligned} s_1 s_2 &= r_1 e^{\theta_1 \imath} r_2 e^{\theta_2 \imath} = r_1 r_2 e^{\theta_1 \imath} e^{\theta_2 \imath} \\ &= r_1 r_2 e^{(\theta_1 + \theta_2)\imath} \end{aligned}$$

$$\begin{aligned} s_1 \cdot s_2 &= (a_1 + b_1 \imath)(a_2 + b_2 \imath) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\imath \end{aligned}$$

For multiplication

- magnitudes and phases *decouple*.
- magnitudes multiply
- phases add



Complex Numbers
Division

For *Division*, the benefit is even greater.

$$s_1 = r_1 e^{\theta_1 \imath}$$
$$s_2 = r_2 e^{\theta_2 \imath}$$

$$s_1 = a_1 + b_1 \imath$$
$$s_2 = a_2 + b_2 \imath$$

$$s_1/s_2 = r_1 e^{\theta_1 \imath} r_2^{-1} e^{-\theta_2 \imath} = \\ = \frac{r_1}{r_2} e^{(\theta_1 - \theta_2)\imath}$$

$$s_1/s_2 = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}\imath$$

For division in polar form,

- Again, magnitudes and phases *decouple*.
- magnitudes divide
- phases subtract

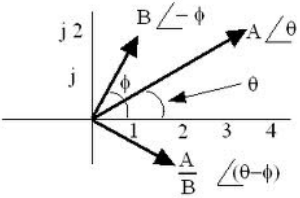


Fig. 17: Division using Polar Form

Complex Numbers

Root Locus

What does this mean for the root locus?
Recall the root locus is the set of s such that

$$1 + k\hat{G}(s) = 0$$

In other words,

$$\hat{G}(s) = -\frac{1}{k}$$

In polar coordinates, this means

$$\hat{G}(s) = \frac{1}{k}e^{\pi \imath}$$

- Magnitude is $1/k$
- Phase is πrad $= 180^\circ$

Since k can be anything greater than 0:

- Root locus is all points such that

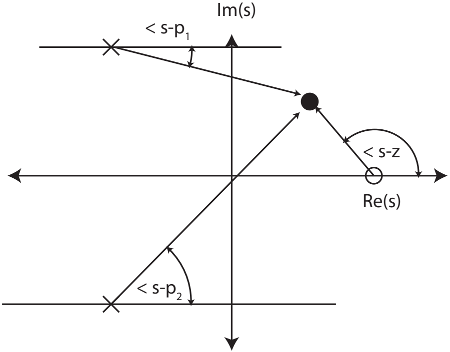
$$\angle\hat{G}(s) = 180^\circ \pm n \cdot 360^\circ$$

Since

$$\hat{G}(s) = \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

Then

$$\begin{aligned}\angle\hat{G}(s) &= \angle(s - z_1) + \cdots + \angle(s - z_m) \\ &\quad - \angle(s - p_1) - \cdots - \angle(s - p_n) \\ &= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i)\end{aligned}$$



For a point on the root locus:

$$\sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = 180^\circ \pm n \cdot 360^\circ$$

Summary

What have we learned today?

Review of Feedback

- Closing the Loop
- Pole Locations

The Effect of Changes in Gain

- Numerical Examples
  - Pole Locations
- Routh-Hurwitz

A Review of Complex Numbers

- Polar Form
- Multiplication-Division

Next Lecture: Constructing the Root Locus