Systems Analysis and Control

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Lecture 14: Root Locus Continued
Overview

In this Lecture, you will learn:

Review: What happens at high gain?
  • Angles of Departure

The Case of $90^\circ$ Departure
  • Calculating the center of asymptotes

Breaking off the Real Axis
  • Break Points

What is the effect of small gain?
  • Departure Angles
Pole locations change at high gain.

- Some poles stay small
- Some poles get large
  - Asymptotes depend on relative number of poles and zeros.

Small poles go to zeros.

Big poles leave on asymptotes:

**Cases:**

- $n - m = 0$ - No Asymptotes
- $n - m = 1$ - Asymptote at $180^\circ$
- $n - m = 2$ - Asymptotes at $\pm 90^\circ$
- $n - m = 3$ - Asymptotes at $180^\circ$, $\pm 60^\circ$
- $n - m = 4$ - Asymptotes at $\pm 45^\circ$ and $\pm 135^\circ$
Recall the suspension system:

\[ G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}. \]

**Count:** 2 zeros, 4 poles.

\[ n - m = 2 \]

\[ \angle_\infty = -90^\circ, -270^\circ \]

2 vertical asymptotes at 90° and 270°.

Poles **MAY** destabilize at large gain. But will they???

- Why these poles?
The Asymptotic Center

Recall

- \( m \) = \# of zeroes
- \( n \) = \# of poles

**Problem 1:** When \( n - m = 2 \).
- Is high gain destabilizing?

**Problem 2:** When \( n - m \geq 2 \).
- Which poles get big?

**Definition 1.**
The **Center of Asymptotes** is where all asymptotes meet.

The center of asymptotes is only for the big poles on the root locus.
- The center of asymptotes is the average of these points as \( k \to \infty \).

\[
\text{center} = \frac{\sum q_i \text{BIG}}{\# i_{\text{BIG}}}
\]
Calculating the Asymptotic Center

\[ \text{center} = \sum \frac{q_{i BIG}}{\# i_{BIG}} \]

Denote

- \( q_i \) are the CLOSED-LOOP poles
  - \( q_i \) are roots of \( d(s) + kn(s) \)
- \( z_i \) are the zeros (open and closed loop)
  - \( z_i \) are roots of \( n(s) \)
- \( p_i \) are the OPEN-LOOP poles
  - \( p_i \) are roots of \( d(s) \)

Calculating \( \#i_{BIG} \) is easy!

- Small poles go to zeroes
- Big poles form asymptotes

\[ \#i_{BIG} = n - m = \#\text{OL poles} - \#\text{OL zeroes} \]

Real Problem: How to calculate \( \sum q_{i BIG} \)?
Calculating the Asymptotic Center

Recall from Routh-Hurwitz: Let $p_i$ be the roots of $d(s)$.

$$d(s) = s^n + a_1 s^{n-1} + \cdots + a_n = (s - p_1)(s - p_2) \cdots (s - p_n)$$

Observe what happens as we expand out the roots:

\[
d(s) = (s - p_1)(s - p_2)(s - p_3)(s - p_4) \cdots (s - p_n) \\
= (s^2 - (p_1 + p_2)s + p_1p_2)(s - p_3)(s - p_4) \cdots (s - p_n) \\
= (s^3 - (p_1 + p_2 + p_3)s^2 + (p_1p_2 + p_2p_3 + p_1p_3)s - p_1p_2p_3)(s - p_4) \cdots (s - p_n) \\
= \cdots \\
= s^n - (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + (-1)^n p_1p_2 \cdots p_n
\]

The second coefficient is the negative sum of the roots

$$a_1 = -(p_1 + p_2 + \cdots + p_n) = - \sum p_i$$
Calculating the Asymptotic Center

Since \( \frac{kG}{1+kG} = \frac{kn}{d+kn} \), \( \sum q_i \) is the second coefficient of \( d(s) + kn(s) \).

Only interested in the case when \( n - m \geq 2 \):

- 90° asymptotes or more.

\[
\begin{align*}
d(s) &= s^n + a_1 s^{n-1} + \cdots \\
n(s) &= s^m + \cdots
\end{align*}
\]

When \( n - m = 2 \),

\[
d(s) + kn(s) = s^n + a_1 s^{n-1} + (a_2 + k)s^{n-2} + \cdots
\]

**Conclusion:** Changing \( k \) doesn’t change the second coefficient.

- Sum of poles doesn’t change under feedback.

\[
\sum p_i = \sum q_i = -a_1
\]

This sum is the second coefficient of \( d(s) \).
Calculating the Asymptotic Center

Recall we want to find

\[
center = \frac{\sum q_{iBIG}}{\#i_{BIG}}
\]

It is obvious that

\[
\sum q_i = \sum q_{iBIG} + \sum q_{iSMALL} = -a_1
\]

So that

\[
\sum q_{iBIG} = -a_1 - \sum q_{iSMALL}
\]

So how do we find \( \sum q_{iSMALL} \)?

- As \( k \to \infty \) small poles go to zeroes.

At high gain

\[
\sum q_{iSMALL} \approx \sum z_i
\]
Calculating the Asymptotic Center

\[ G(s) = \frac{n(s)}{d(s)} \]

The zeros, \( z_i \) are the roots of \( n(s) \).

\[ n(s) = s^m + b_1 s^{m-1} + \cdots = (s - z_1) \cdots (s - z_m) \]

As before

\[ \sum z_i = -b_1 \]

Finally

\[ \text{center} = \frac{\sum q_i_{\text{BIG}}}{\#i_{\text{BIG}}} = \frac{-a_1 - \sum q_i_{\text{SMALL}}}{n - m} \]

\[ \Rightarrow \quad \frac{-a_1 - \sum z_i}{n - m} = \frac{b_1 - a_1}{n - m} \]

Where

- \( a_1 \) is the first coefficient of \( d(s) \)
- \( b_1 \) is the first coefficient of \( n(s) \)
Calculating the Asymptotic Center

Example: Suspension System

\[ G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1} \]

\[ \#i_{BIG} = n - m \]
\[ = \#poles - \#zeroes \]
\[ = 2. \]

Read off the coefficients

- \( a_1 = 2 \)
- \( b_1 = 1 \)

\[ \text{center} = \frac{b_1 - a_1}{n - m} = \frac{1 - 2}{2} = -\frac{1}{2} \]

Conclusion: High gain is stable.
Calculating the Asymptotic Center

Example: Tweaked Suspension System

Look what happens if we change 2\textsuperscript{nd} coefficient in \(n(s)\) from 1 to 3.

\[
G(s) = \frac{s^2 + 3s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}
\]

\(#_{BIG} = n - m = \#\text{poles} - \#\text{zeroes} = 2\)

Read off the coefficients

- \(a_1 = 2\)
- \(b_1 = 3\)

Thus

\[
\text{center} = \frac{b_1 - a_1}{n - m} = \frac{3 - 2}{2} = \frac{1}{2}
\]

Now high gain is unstable.
Calculating the Asymptotic Center

Example: Suspension System with Integral Feedback

\[ G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1} \cdot \frac{1}{s} = \frac{s^2 + s + 1}{s^5 + 2s^4 + 3s^3 + 1s^2 + s} \]

\[#i_{BIG} = n - m = \#poles - \#zeroes = 3.\]

Again, we have the same coefficients

- \(a_1 = 2\)
- \(b_1 = 1\)

Thus

\[center = \frac{b_1 - a_1}{n - m} = \frac{1 - 2}{3} = -\frac{1}{3}\]
Calculating the Asymptotic Center

Another Example

\[ G(s) = \frac{s^2 + s + 1}{s^6 + 2s^5 + 5s^4 - s^3 + 2s^2 + 1} \]

First, \( \#_{i_{BIG}} = n - m = 4 \).

Again, we have the same coefficients

- \( a_1 = 2 \)
- \( b_1 = 1 \)

Thus

\[ \text{center} = \frac{b_1 - a_1}{n - m} = \frac{1 - 2}{4} = -\frac{1}{4} \]
Calculating the Asymptotic Center
Using poles and zeros directly

Expand the formula for asymptotic center:

$$\text{center} = \frac{b_1 - a_1}{n - m} = \sum p_i - \sum z_i$$

If we know the $p_i$ and $z_i$, we can use these instead of the $a_1$ and $b_1$

$$G(s) = \frac{1}{s(s + 4)(s + 6)}$$

First, $\#_{i_{BIG}} = n - m = 3$

This time, we directly use poles and zeros

- **No Zeros**
- $p_1 = 0$, $p_2 = -4$, $p_3 = -6$.

$$\sum p_i = -4 - 6 = -10$$

$$\text{center} = \frac{\sum p_i - \sum z_i}{n - m} = \frac{-10 - 0}{3} = -3.33\overline{3}$$
Calculating the Asymptotic Center

DIY Example

\[ G(s) = \frac{s + 2}{(s + 1)(s^2 + 2s + 2)} \]
Break points

Recall the inverted pendulum with derivative feedback.

\[ G(s) = \frac{1 + s}{s^2 - \frac{1}{2}} \]

When do the poles become imaginary?

- Important for choosing optimal \( k \).
Break points

Other Examples

\[ G(s) = \frac{(s + 3)(s + 4)}{(s + 1)(s + 2)} \]
Break points

Recall for a point on the root locus

\[ d(s) + kn(s) = 0 \]

or for a point on the real axis: \( s = a \)

\[ k(a) = -\frac{d(a)}{n(a)} = -\frac{1}{G(a)} \]

**Idea:** Use maximum principle to find the maximum and minimum of \( k \) on the real axis.

**Definition 2.**

The extrema of a continuous function of a real variable, \( f(a) \), occur at the boundary or when

\[ \frac{d}{da} f(a) = 0 \]
To find the point when the root locus leaves the real axis, we calculate the extrema of

\[ k(a) = -\frac{1}{G(a)} \]

We need to solve

\[ \frac{d}{da} k(a) = 0 \]

or

\[ \frac{d}{da} k(a) = -\frac{d}{da} \frac{1}{G(a)} = \frac{d(a)}{n(a)^2} n'(a) - \frac{d'(a)}{n(a)} = \frac{d(a)n'(a) - d'(a)n(a)}{n(a)^2} = 0 \]

Break Points occur at real-valued solutions of

\[ d(a)n'(a) - d'(a)n(a) = 0 \]
Break points

**Numerical Example**

\[ G(s) = \frac{1}{s(s + 4)(s + 6)} = \frac{1}{s^3 + 10s^2 + 24s} \]

Break points occur when

\[
d(a)n'(a) - d'(a)n(a) = 0 - (3a^2 + 20a + 24) = 0
\]

which has roots

\[
a_{1,2} = \frac{-20 \pm \sqrt{20^2 - 4 \times 24 \times 3}}{6} \\
\approx -5.1, -1.57
\]
Break points
Numerical Example

\[ G(s) = \frac{(s + 3)(s + 4)}{(s + 1)(s + 2)} = \frac{s^2 + 7s + 12}{s^2 + 3s + 2} \]

Break points occur when

\[ d(a)n'(a) - d'(a)n(a) = (a^2 + 3a + 2)(2a + 7) - (2a + 3)(a^2 + 7a + 12) \]
\[ = (a^2 + 3a + 2)(2a + 7) - (2a + 3)(a^2 + 7a + 12) \]
\[ = -2(2a^2 + 10a + 11) = 0 \]

Which has roots

\[ a_{1,2} = -1.634, -3.366 \]

Break points at −1.634 and −3.366.
Break points
Numerical Example

\[ G(s) = \frac{1 + s}{s^2 - \frac{1}{2}} \]

Break points occur when

\[ d(a)n'(a) - d'(a)n(a) \]
\[ = (a^2 - .5) \cdot 1 - 2a \cdot (1 + a) \]
\[ = -(a^2 + 2a + .5) = 0 \]

Which has roots

\[ a_{1,2} = -0.293, -1.707 \]

Break points at \(-0.293\) and \(-1.707\)
**Step 1:** Root Locus starts at Open Loop Poles.

**Step 2:** At Large Gain, $k \to \infty$
- Small Poles go to zeroes
- Large Poles approach asymptotes
- Center at

$$\sigma = \frac{\sum p_i - \sum z_i}{n - m} = \frac{b_1 - a_1}{n - m}$$

**Step 3:** On real axis
- When odd number of poles/zeroes to the right.
- Break points when

$$-\frac{d}{da} \frac{1}{G'(a)} = 0 \quad \text{or} \quad d(a)n'(a) - d'(a)n(a) = 0$$
Departure Angle

The root locus starts at the poles.

- What is the effect of small gain?
- Do the poles become more or less stable?

![Root Locus Diagram]

\[ \text{Im}(s) \quad \text{Re}(s) \]

\[ -3.5 \quad -3 \quad -2.5 \quad -2 \quad -1.5 \quad -1 \quad 0 \quad 0.5 \quad 1 \quad 1.5 \]

\[ -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ -8 \quad -6 \quad -4 \quad -2 \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \]
To find the departure angle, we look at a very small region around the departure point.

For a point to be on the root locus, we want phase of $180^\circ$.

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = 180^\circ$$
Departure Angle

If we make the point \( s \) extremely close to the pole \( p \).

- The angle to other poles and zeros from \( s \) is the same as from \( p \).
  - \( \angle (s - z_i) \approx \angle (p - z_i) \) for all \( i \)
  - \( \angle (s - p_i) \approx \angle (p - p_i) \) for all \( i \)
- The only difference is the phase from \( p \) itself.

The phase due to \( p \) equals the departure angle,

\[
\angle_{dep} \quad \angle (s - p) = \angle_{dep}
\]

The total phase is

\[
\angle G(s) \cong \angle G(p) - \angle (s - p) = \angle G(p) - \angle_{dep} = 180^\circ
\]

Thus the departure angle from pole \( p \) is

\[
\angle_{dep} = \angle G(p) + 180^\circ
\]

Therefore, to find the departure angle from pole \( p \), just find the phase at \( p \).
The phase at $p$ is based on geometry.

$$\angle G(p) = 150^\circ - 90^\circ - 45^\circ = 15^\circ$$

So the departure angle is easy to calculate.

$$\angle_{dep} = \angle G(p) + 180^\circ = 195^\circ$$
Departure Angle
Numerical Examples

\[ G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1} \]

Poles at
- \[ p_{1,2} = -0.957 \pm 1.23i \]
- \[ p_{3,4} = -0.0433 \pm 0.641i \]

Zeroes at
- \[ z_{1,2} = -0.5 \pm 0.866i \]

Problem:
Find departure angle at \( p_1 = -0.957 + 1.23i \).

\[ \angle_{dep} = 180^\circ + \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4) \]

The difficulty is calculating the phase.
\[ \angle(p_1 - z_1) = \angle(-0.957 + 1.23i + 0.5 - 0.866i) \]
\[ = \angle(-0.457 + 0.364i) \]
\[ = \tan^{-1}\left(\frac{0.364}{-0.457}\right) \]
\[ = 141.46^\circ \]

\[ \angle(p_1 - z_2) = \angle(-0.457 + 2.096i) = 102.3^\circ \]

Obviously,

\[ \angle(p_1 - p_2) = 90^\circ \]
\[ \angle(p_1 - p_3) = 147.2^\circ , \quad \angle(p_1 - p_4) = 116.03^\circ \]
Now that we have all the angles:

\[ \angle G(p_1) = \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4) \]
\[ = 141.46^\circ + 102.3^\circ - 90^\circ - 147.2^\circ - 116.03^\circ \]
\[ = -109.47^\circ \]

We conclude

\[ \angle_{dep,p_1} = \angle G(p_1) + 180^\circ = 70.53^\circ \]

By symmetry we could find

\[ \angle_{dep,p_2} = -70.53^\circ \]
What about a pole on the real axis?

\[ \angle G(p) = 0° \text{ or } 180° \]
Calculating the Departure Angle

DIY Example

\[ G(s) = \frac{s + 2}{(s + 1)(s^2 + 2s + 2)} \]
Summary

What have we learned today?

Review: What happens at high gain?
  • Angles of Departure

The Case of $90^\circ$ Departure
  • Calculating the center of asymptotes

Breaking off the Real Axis
  • Break Points

What is the effect of small gain?
  • Departure Angles

Next Lecture: Arrival Angles, Summary + Examples