Systems Analysis and Control

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Lecture 18: The Frequency Response
In this Lecture, you will learn:

**Introduction to the Frequency Domain**
- Life without Newton
  - “Who needs a model, anyway?”
- Black Boxes.

**Frequency Response**
- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots
Definition 1.

The **Frequency Response** is the *steady-state* output of a system with sinusoidal input.

Figure: Response of Concrete Slabs to Soil Excitation (FEM)
The Frequency Response

A Sinusoidal Input:

\[ u(t) = A \sin(\omega t) + B \cos(\omega t) \]
\[ = \sqrt{A^2 + B^2} \sin \left( \omega t - \tan^{-1} \left( \frac{B}{A} \right) \right) \]
\[ = M \sin(\omega t + \phi) \]

Laplace Transform:

\[ \hat{u}(s) = \frac{Bs + A\omega}{s^2 + \omega^2} \]

- \[ M = \sqrt{A^2 + B^2} \]
- \[ \phi = -\tan^{-1} \left( \frac{B}{A} \right) \]
The Frequency Response

For now, set $B = 0$, then $u(t) = A \sin \omega t$.

\[ \hat{u}(s) = \frac{A\omega}{s^2 + \omega^2} \]

For a given stable transfer function,

\[ G(s) = \frac{n(s)}{(s + p_1) \cdots (s + p_n)} \]

then by partial-fraction expansion

\[ \hat{y}(s) = G(s) \hat{u}(s) = G(s) \frac{A\omega}{(s + \omega)(s - \omega)} \]

\[ = \frac{r_1}{s + p_1} + \cdots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + \omega} + \frac{\beta}{s - \omega} . \]
The Frequency Response

Partial Fraction Expansion:

\[ \hat{y}(s) = \frac{r_1}{s + p_1} + \cdots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + \omega} + \frac{\beta}{s - \omega} \]

Inverse Laplace Transform:

\[ y(t) = r_1 e^{-p_1 t} + \cdots + r_n e^{-p_n t} + \alpha e^{-\omega t} + \beta e^{\omega t} \]

But we want the **Steady-State Response**.

- Poles \( p_i \) are all stable.
  - \( \lim_{t \to \infty} e^{-p_i t} = 0 \)
- These are called **Transient Responses**
- only left with

\[ y_{ss}(t) = \alpha e^{-\omega t} + \beta e^{\omega t} \]
The Frequency Response

Since $\pm \omega$ are isolated poles, by the remainder theorem:

\[
\alpha = G(s) \frac{A\omega}{(s + i\omega)(s - i\omega)(s + \omega)} \bigg|_{s = -i\omega}
\]

\[
= G(-\omega) \frac{A\omega}{-2\omega}
\]

\[
= G(-\omega) \frac{A}{-2i}
\]

Likewise,

\[
\beta = G(i\omega) \frac{A}{2i}
\]

Then

\[
y_{ss}(t) = \alpha e^{-\omega t} + \beta e^{\omega t}
\]

\[
= A \frac{G(i\omega)e^{\omega t} - G(-\omega)e^{-\omega t}}{2i}
\]
**Issue:** \( G(-\omega) \) is the complex conjugate of \( G(\omega) \).

**Definition 2.**

For a complex number \( s = a + bi \), the **Complex Conjugate** of \( s \) is

\[
s^* = a - bi
\]

- Just replace \( i \rightarrow -i \).
- \( re^{i\theta} \rightarrow re^{-i\theta} \)

Magnitude is unchanged. Phase is reversed.

For \( s = re^{i\theta} \),

**Phase:** \( \angle s = \theta \)

- \( \angle s^* = -\theta = -\angle s \)

**Magnitude:** \( |s^*| = r = |s| \)
The Frequency Response

Complex Conjugate: $G(-\omega) = G(\omega)^*$

$$y_{ss}(t) = A \frac{G(\omega)e^{i\omega t} - G(-\omega)e^{-i\omega t}}{2i}$$

Recall that we can express $G(\omega)$ as

$$G(\omega) = |G(\omega)|e^{\angle G(\omega)i}$$

and $|G(\omega)| = |G(\omega)^*| = |G(-\omega)|$, $\angle G(-\omega) = \angle G(-\omega)^* = -\angle G(\omega)$

$$y_{ss}(t) = A \frac{G(\omega)e^{i\omega t} - G(-\omega)e^{-i\omega t}}{2i}$$

$$= |G(\omega)|A \frac{e^{\angle G(\omega)i}e^{i\omega t} - e^{-\angle G(\omega)i}e^{-i\omega t}}{2i}$$

$$= |G(\omega)|A \frac{e^{(\omega t + \angle G(\omega))i} - e^{-(\omega t + \angle G(\omega))i}}{2i}$$

$$= |G(\omega)|A \sin(\omega t + \angle G(\omega))$$
If the input is shifted:

**Input:**

\[ u(t) = M \sin(\omega t + \phi) \]

**Output:**

\[ y(t) = M |G(i\omega)| \sin(\omega t + \phi + \angle G(i\omega)) \]
**Conclusion:** The response to a sinusoidal input $\sin \omega t$:

- A sinusoid with the same frequency.
- *Phase* is shifted by $\angle G(\omega)$.
- *Magnitude* is changed $|G(\omega)|$.

We refer to

- $|G(\omega)|$ is the **Magnitude** of Frequency Response
- $\angle G(\omega)$ is the **Phase** of Frequency Response

These depend only on $\omega$ and $G(\omega)$. 
The amplification at the natural frequency, $\omega_n$, is called resonance.

**Figure**: Frequency Sweeping with Resonance
**Application:** Crane Oscillation

- Sinusoidal Input from Hanging load.
- Avoid Spillage.
- Avoid Tipping.

A Form of **Motion Control**.
Frequency Response Planning

Applications

Figure: Simple Crane Sway Control

Figure: Industrial Crane Sway Control

Figure: Failure of Crane Control
**Application:** Building Response to Earthquakes

- Sinusoidal input from ground.
- Reduce peak output.
Obtaining Frequency Response Data
Controlling Structural Dynamics

Figure: Earthquake Damping
The Frequency Response

This can work the other way too:

- **Input** $u(t) \sin \omega t$
- **Output**: $y(t) = M \sin(\omega t + \phi)$
- **Measure** $M$ and $\phi$
  - Relative Phase $\phi = \angle G(\omega)$
  - Magnitude: $M = |G(\omega)|$

**Frequency Sweeping**: Measure $M$ and $\phi$ at every frequency

- Get functions $M(\omega)$ and $\phi(\omega)$

Reconstruct

$$G(s) \approx M(s)e^{\phi(s)i}$$
The Frequency Response

Input: A Sinusoid of Increasing Frequency.

\[ u(t) = \sin ((\omega_0 + kt)t) \]

Graph showing the sinusoidal input signal over a time span from 0 to 5 seconds.
Complex Poles and Zeros

Figure: Frequency Sweeping with Resonance
The Frequency Response

Figure: A Frequency Sweep in Circuit Analysis
Frequency Sweeping
Magnitude and Phase Data
Magnitude and Phase Data for a Notch Filter

This type of Magnitude-Phase graph is called a Bode Plot
Frequency Sweeping
Magnitude and Phase Data

Figure: Data From a Graphic Equalizer

No Model is Required to understand the system.
Obtaining Frequency Response Data
Finite-Element Modeling

For structures and rigid bodies.

- Dynamics are Partial-Differential Equations
  - Elasticity
- We can derive the model, but it would be too complicated.

We must rely on Simulation.

- Simulate a sinusoidal input
  - Record output displacement
- Resulting model is only an approximation.
Figure: Satellite Frequency Response Analysis using NASTRAN
Summary

What have we learned today?

Introduction to the Frequency Domain

- Life without Newton
  - “Who needs a model, anyway?”
- Black Boxes.

Frequency Response

- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots

Next Lecture: The Bode Plot
Obtaining Frequency Response Data

Experimental Methods: Circuit Sweeping

Figure: Frequency Response Analysis in the Power Industry (Ad)