Systems Analysis and Control

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Lecture 18: The Frequency Response

Overview

In this Lecture, you will learn:

Introduction to the Frequency Domain

- Life without Newton
  - "Who needs a model, anyway?"
- Black Boxes.

Frequency Response

- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots

The Frequency Response

Definition 1.

The **Frequency Response** is the *steady-state* output of a system with sinusoidal input.

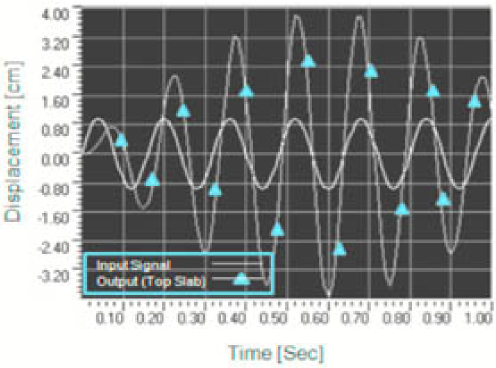


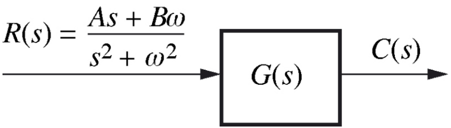
Figure : Response of Concrete Slabs to Soil Excitation (FEM)

A Sinusoidal Input:

$$\begin{aligned} u(t) &= A\sin(\omega t) + B\cos(\omega t) \\ &= \sqrt{A^2+B^2}\sin\left(\omega t - \tan^{-1}\left(\frac{B}{A}\right)\right) \\ &= M\sin(\omega t + \phi) \end{aligned}$$

Laplace Transform:

$$\hat{u}(s) = \frac{Bs + A\omega}{s^2+\omega^2}$$

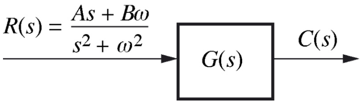


- $M = \sqrt{A^2+B^2}$
- $\phi = -\tan^{-1}\left(\frac{B}{A}\right)$

The Frequency Response

For now, set $B = 0$, then $u(t) = A \sin \omega t$.

$$\hat{u}(s) = \frac{A\omega}{s^2 + \omega^2}$$



For a given stable transfer function,

$$G(s) = \frac{n(s)}{(s + p_1) \cdots (s + p_n)},$$

then by partial-fraction expansion

$$\begin{aligned}\hat{y}(s) &= G(s)\hat{u}(s) \\ &= G(s)\frac{A\omega}{(s + \imath\omega)(s - \imath\omega)} \\ &= \frac{r_1}{s + p_1} + \cdots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + \imath\omega} + \frac{\beta}{s - \imath\omega}.\end{aligned}$$

The Frequency Response

Partial Fraction Expansion:

$$\hat{y}(s) = \frac{r_1}{s+p_1} + \cdots + \frac{r_n}{s+p_n} + \frac{\alpha}{s+\imath\omega} + \frac{\beta}{s-\imath\omega}$$

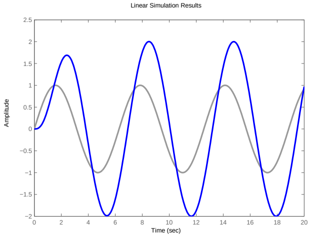
Inverse Laplace Transform:

$$y(t) = r_1 e^{-p_1 t} + \cdots + r_n e^{-p_n t} \; + \; \alpha e^{-\imath\omega t} + \beta e^{\imath\omega t}$$

But we want the **Steady-State Response**.

- Poles p_i are all stable.
  - $\lim_{t\to\infty} e^{-p_i t} = 0$
- These are called **Transient Responses**
- only left with

$$y_{ss}(t) = \alpha e^{-\imath\omega t} + \beta e^{\imath\omega t}$$



The Frequency Response

Since $\pm\imath\omega$ are isolated poles, by the remainder theorem:

$$\begin{aligned}\alpha &= G(s)\frac{A\omega}{(s+\imath\omega)(s-\imath\omega)}(s+\imath\omega)|_{s=-\imath\omega}\\ &= G(-\imath\omega)\frac{A\omega}{-2\imath\omega}\\ &= G(-\imath\omega)\frac{A}{-2\imath}\end{aligned}$$

Likewise,

$$\beta = G(\imath\omega)\frac{A}{2\imath}$$

Then

$$\begin{aligned}y_{ss}(t) &= \alpha e^{-\imath\omega t} + \beta e^{\imath\omega t}\\ &= A\frac{G(\imath\omega)e^{\imath\omega t} - G(-\imath\omega)e^{-\imath\omega t}}{2\imath}\end{aligned}$$

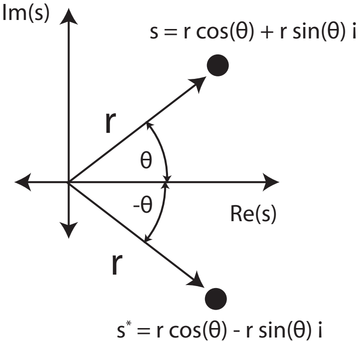
Issue: $G(-\imath\omega)$ is the complex conjugate of $G(\imath\omega)$.

Definition 2.

For a complex number $s = a + b\imath$, the **Complex Conjugate** of s is

$$s^* = a - b\imath$$

- Just replace $\imath \to -\imath$.
- $re^{\imath\theta} \to re^{-\imath\theta}$



Magnitude is unchanged. Phase is reversed

For $s = re^{\imath\theta}$,

Phase: $\angle s = \theta$

- $\angle s^* = -\theta = -\angle s$

Magnitude: $|s| = r$

- $|s^*| = r = |s|$

The Frequency Response

Complex Conjugate: $G(-\imath\omega) = G(\imath\omega)^*$

$$y_{ss}(t) = A\frac{G(\imath\omega)e^{\imath\omega t} - G(-\imath\omega)e^{-\imath\omega t}}{2\imath}$$

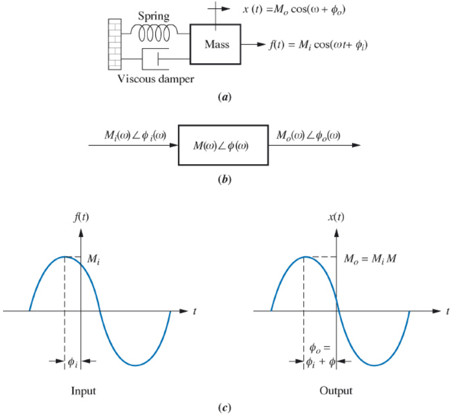
Recall that we can express $G(\imath\omega)$ as

$$G(\imath\omega) = |G(\imath\omega)|e^{\angle G(\imath\omega)\imath}$$

and $|G(\imath\omega)| = |G(\imath\omega)^*| = |G(-\imath\omega)|$, $\angle G(-\imath\omega) = \angle G(-\imath\omega)^* = -\angle G(\imath\omega)$

$$\begin{aligned}
y_{ss}(t) &= A\frac{G(\imath\omega)e^{\imath\omega t} - G(-\imath\omega)e^{-\imath\omega t}}{2\imath} \\
&= |G(\imath\omega)|A\frac{e^{\angle G(\imath\omega)}e^{\imath\omega t} - e^{-\angle G(\imath\omega)}e^{-\imath\omega t}}{2\imath} \\
&= |G(\imath\omega)|A\frac{e^{(\omega t+\angle G(\imath\omega))\imath} - e^{-(\omega t+\angle G(\imath\omega))\imath}}{2\imath} \\
&= |G(\imath\omega)|A\sin(\omega t + \angle G(\imath\omega))
\end{aligned}$$

The Frequency Response



If the input is shifted:

Input:

$$u(t) = M \sin(\omega t + \phi)$$

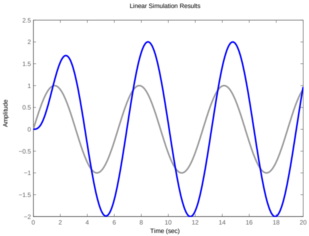
Output:

$$y(t) = M|G(\imath\omega)| \sin(\omega t + \phi + \angle G(\imath\omega))$$

The Frequency Response

Conclusion: The response to a sinusoidal input $\sin \omega t$:

- A sinusoid with the same frequency.
- *Phase* is shifted by $\angle G(\imath\omega)$.
- *Magnitude* is changed $|G(\imath\omega)|$.



We refer to

- $|G(\imath\omega)|$ is the Magnitude of Frequency Response
- $\angle G(\imath\omega)$ is the Phase of Frequency Response

These depend only on ω and $G(\imath\omega)$.

Complex Poles and Zeros

The amplification at the natural frequency, ω_n, is called resonance.

Figure : Frequency Sweeping with Resonance

Application: Crane Oscillation

- Sinusoidal Input from Hanging load.
- Avoid Spillage.
- Avoid Tipping.

A Form of **Motion Control**.

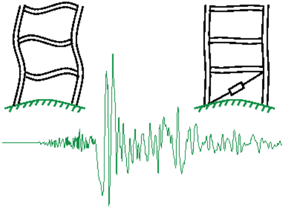
Figure : Simple Crane Sway Control

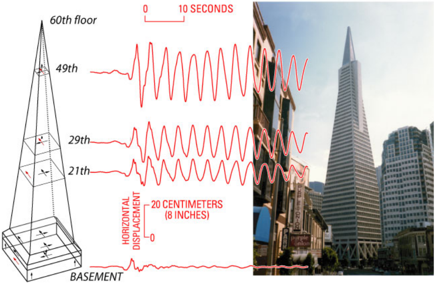
Figure : Industrial Crane Sway Control

Figure : Failure of Crane Control

Application: Building Response to Earthquakes

- Sinusoidal input from ground.
- Reduce peak output.





Obtaining Frequency Response Data

Controlling Structural Dynamics

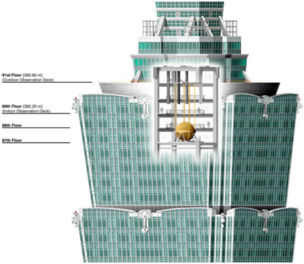
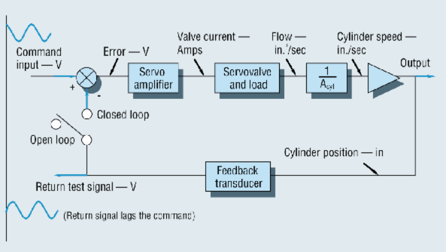


Figure : Earthquake Damping

The Frequency Response

This can work the other way too:

- Input $u(t)\sin\omega t$
- Output: $y(t) = M\sin(\omega t + \phi)$
- Measure M and ϕ
  - Relative Phase $\phi = \angle G(\imath\omega)$
  - Magnitude: $M = |G(\imath\omega)|$

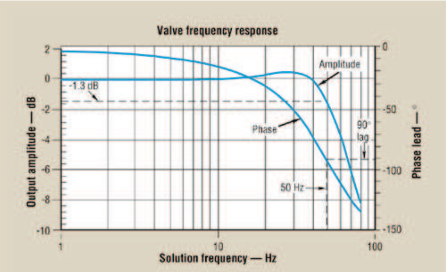


Frequency Sweeping: Measure M and ϕ at every frequency

- Get functions $M(\omega)$ and $\phi(\omega)$

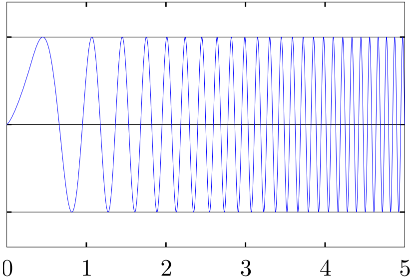
Reconstruct

$$G(s) \cong M(s)e^{\phi(s)\imath}$$



Input: A Sinusoid of Increasing Frequency.

$$u(t) = \sin\left((\omega_0 + kt)t\right)$$

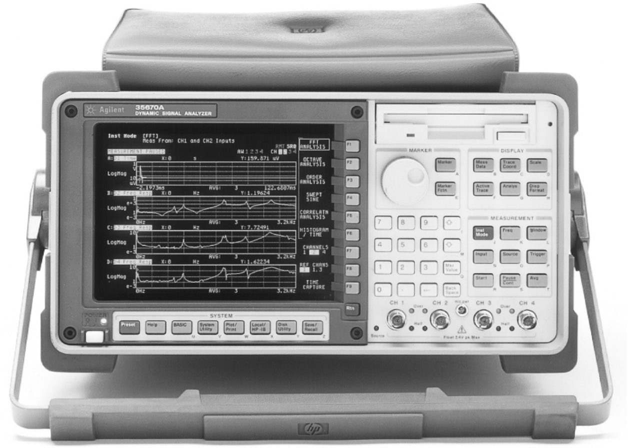


Complex Poles and Zeros

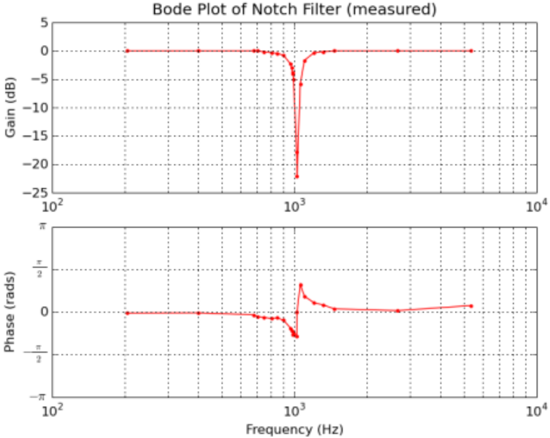
Figure : Frequency Sweeping with Resonance

The Frequency Response

Figure : A Frequency Sweep in Circuit Analysis


Agilent
35670A
DYNAMIC SIGNAL ANALYZER
FFT ANALYSIS
OCTAVE ANALYSIS
ORDER ANALYSIS
SWEPT SINE
CORRELATN ANALYSIS
HISTOGRAM / TIME
CHANNELS
REF CHANS
TIME CAPTURE
MARKER
DISPLAY
MEASUREMENT
SYSTEM

Magnitude and Phase Data for a Notch Filter



This type of Magnitude-Phase graph is called a Bode Plot

Frequency Sweeping

Magnitude and Phase Data

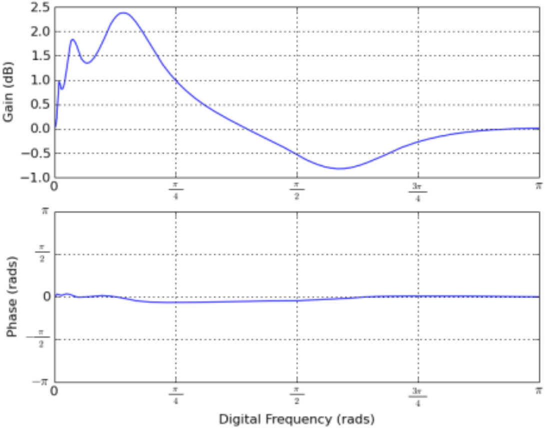


Figure : Data From a Graphic Equalizer

No Model is Required to understand the system.

Obtaining Frequency Response Data

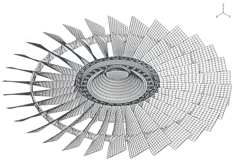
Finite-Element Modeling

For structures and rigid bodies.

- Dynamics are Partial-Differential Equations
  - Elasticity
- We can derive the model, but it would be too complicated.

We must rely on Simulation.

- Simulate a sinusoidal input
  - Record output displacement
- Resulting model is only an approximation.



Obtaining Frequency Response Data

Finite-Element Modeling

Figure : Satellite Frequency Response Analysis using NASTRAN

Summary

What have we learned today?

Introduction to the Frequency Domain

- Life without Newton
  - "Who needs a model, anyway?"
- Black Boxes.

Frequency Response

- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots

Next Lecture: The Bode Plot

Obtaining Frequency Response Data

Experimental Methods: Circuit Sweeping

Figure : Frequency Response Analysis in the Power Industry (Ad)