

Systems Analysis and Control

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Lecture 20: Drawing Bode Plots, Part 2

In this Lecture, you will learn:

Simple Plots

- Real Zeros
- Real Poles
- Complex Zeros
- Complex Poles

Review

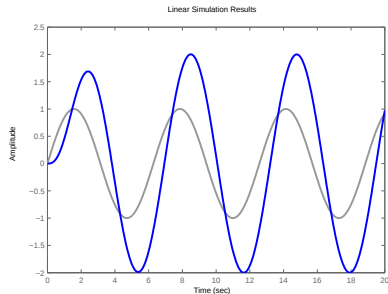
Recall: Frequency Response

Input:

$$u(t) = M \sin(\omega t + \phi)$$

Output: Magnitude and Phase Shift

$$y(t) = |G(j\omega)| M \sin(\omega t + \phi + \angle G(j\omega))$$



Frequency Response to $\sin \omega t$ is given by $G(j\omega)$

Review

Recall: Bode Plot

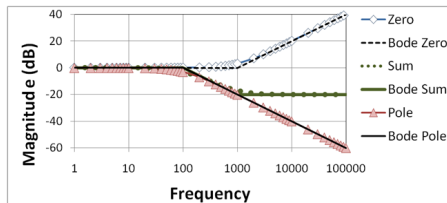
Definition 1.

The **Bode Plot** is a pair of log-log and semi-log plots:

1. Magnitude Plot: $20 \log_{10} |G(j\omega)|$ vs. $\log_{10} \omega$
2. Phase Plot: $\angle G(j\omega)$ vs. $\log_{10} \omega$

Bite-Size Chunks:

$$\begin{aligned} \angle G(j\omega) \\ = \sum_i \angle(j\omega\tau_{zi} + 1) - \sum_i \angle(j\omega\tau_{pi} + 1) \end{aligned}$$



$$20 \log |G(j\omega)| = \sum_i 20 \log |j\omega\tau_{zi} + 1| - \sum_i 20 \log |j\omega\tau_{pi} + 1|$$

Plotting Simple Terms

Plotting Normal Zeros

$$G_1(s) = (\tau s + 1)$$

$$|\omega\tau + 1| \cong \begin{cases} 1 & \omega \ll \frac{1}{\tau} \\ |\omega\tau| & \omega \gg \frac{1}{\tau} \end{cases}$$

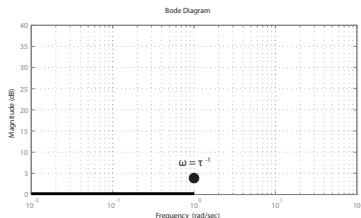
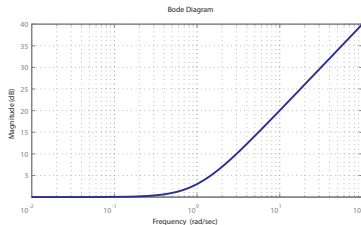
Behaves as

- A constant at low ω .
- A zero at high ω .

Break Point at $\omega = \frac{1}{\tau}$.

Magnitude:

$$20 \log |\omega\tau + 1| = \begin{cases} 0 \text{ dB} & \omega \ll \frac{1}{\tau} \\ 3.01 \text{ dB} & \omega = \frac{1}{\tau} \\ 20 \log \omega & \omega \gg \frac{1}{\tau} \end{cases}$$



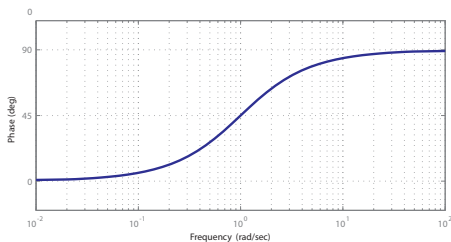
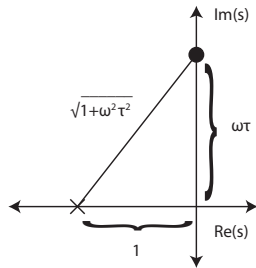
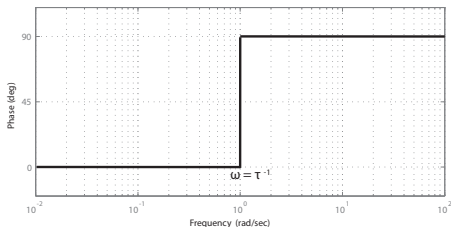
Plotting Real Zeros

Phase

Geometry:

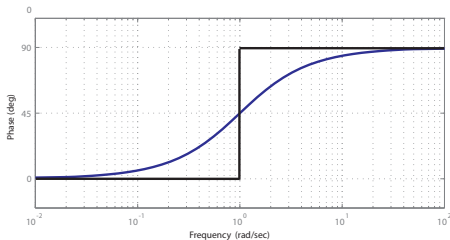
$$\angle(i\omega\tau + 1) = \tan^{-1}\left(\frac{\omega\tau}{1}\right)$$

$$\angle(i\omega\tau + 1) = \begin{cases} 0^\circ & \omega \ll \frac{1}{\tau} \\ 45^\circ & \omega = \frac{1}{\tau} \\ 90^\circ & \omega \gg \frac{1}{\tau} \end{cases}$$



Plotting Real Zeros

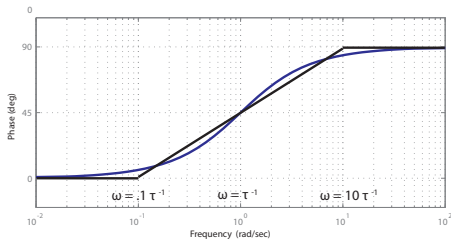
Phase



We can improve our sketch using a line

$$\angle(j\omega\tau + 1)$$

$$= \begin{cases} 0^\circ & \omega < \frac{.1}{\tau} \\ 45^\circ (1 + \log \omega\tau) & \frac{.1}{\tau} < \omega < \frac{10}{\tau} \\ 90^\circ & \omega > \frac{10}{\tau} \end{cases}$$



Plotting Real Poles

Magnitude

We can plot real zeros

- Now lets do the poles.

Poles are the opposite of zeros

$$G_{zero} = G_{pole}^{-1}$$

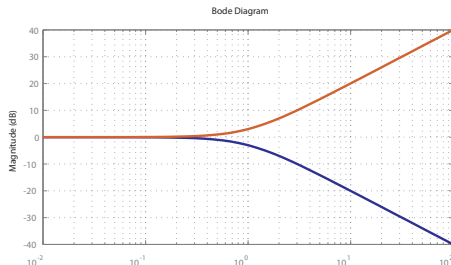
$$G_{pole}(\omega) = \frac{1}{j\omega\tau + 1}$$

$$G_{zero}(\omega) = j\omega\tau + 1$$

Magnitude:

$$|G_{pole}(j\omega)| = \frac{1}{|j\omega\tau + 1|}$$

$$20 \log |G_{pole}(j\omega)| = -20 \log |j\omega\tau + 1|$$

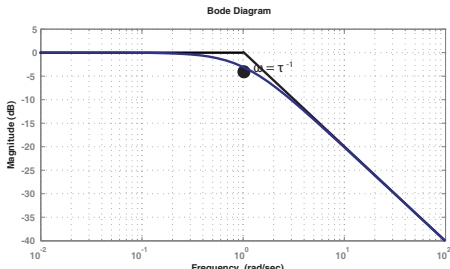
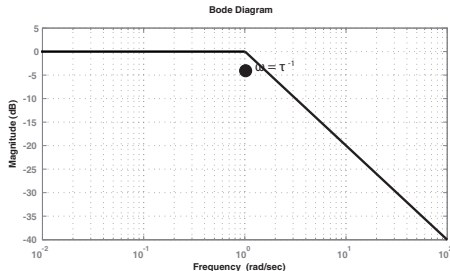


Plotting Real Poles

Magnitude

Our approximation is also a reflection of the zero

$$20 \log \frac{1}{|j\omega\tau + 1|} = \begin{cases} 0 \text{ dB} & \omega \ll \frac{1}{\tau} \\ -3.01 \text{ dB} & \omega = \frac{1}{\tau} \\ -20 \log \omega & \omega \gg \frac{1}{\tau} \end{cases}$$



Plotting Real Poles

Phase

We can plot real zeros

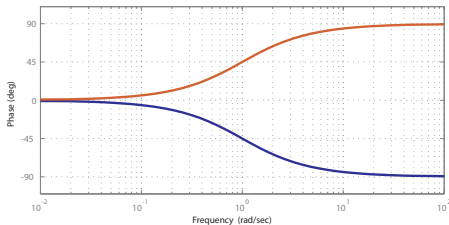
- Now let's do the poles.

Poles are the opposite of zeros

Phase:

$$\begin{aligned}\angle G_{pole}(i\omega) &= -\tan^{-1} \omega\tau \\ &= -\angle G_{zero}(i\omega)\end{aligned}$$

$$G_{pole}(i\omega) = \frac{1}{i\omega\tau + 1} = G_{zero}^{-1}(i\omega)$$



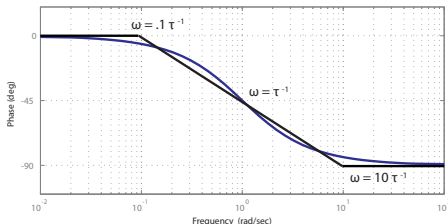
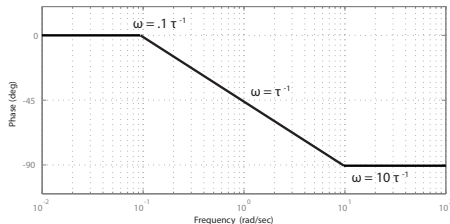
Plotting Real Poles

Phase

Our approximation is also a reflection of the zero

$$\angle(\omega\tau + 1)^{-1}$$

$$= \begin{cases} 0^\circ & \omega < \frac{.1}{\tau} \\ -45^\circ (1 + \log \omega\tau) & \frac{.1}{\tau} < \omega < \frac{10}{\tau} \\ -90^\circ & \omega > \frac{10}{\tau} \end{cases}$$



Plotting Real Poles and Zeros

Lead-Lag

We now have the pieces for a **Lead-Lag Compensator**.

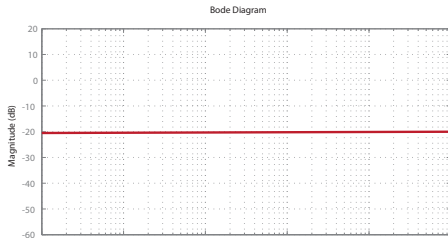
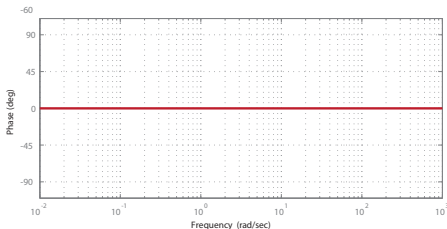
Example: Lead Compensator

$$G(s) = \frac{s + 1}{s + 10}$$

First Step: Standard Form

$$G(s) = \frac{s + 1}{s + 10} = \frac{1}{10} \frac{s + 1}{\frac{1}{10}s + 1}$$

Second Step: Plot the Constant $c = .1 = -20\text{dB}$.



Plotting Real Poles and Zeros

Lead-Lag

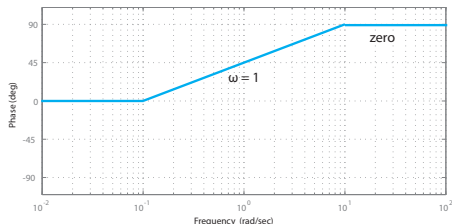
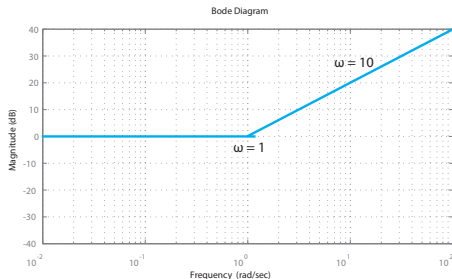
Third Step: Plot the Zero at $\omega = 1$.

Magnitude:

$$20 \log |\omega + 1| = \begin{cases} 0 \text{ dB} & \omega \ll 1 \\ 3.01 \text{ dB} & \omega = 1 \\ 20 \log \omega & \omega \gg 1 \end{cases}$$

Phase:

$$\begin{aligned} & \angle(\omega + 1) \\ &= \begin{cases} 0^\circ & \omega < .1 \\ 45^\circ (1 + \log \omega) & .1 < \omega < 10 \\ 90^\circ & \omega > 10 \end{cases} \end{aligned}$$



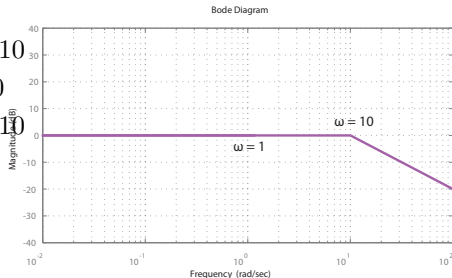
Plotting Real Poles and Zeros

Lead-Lag

Fourth Step: Plot the Pole at $\omega = 10$.

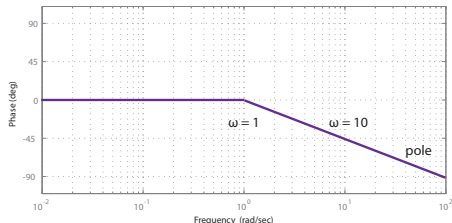
Magnitude:

$$20 \log \left| \frac{j\omega}{10} + 1 \right|^{-1} = \begin{cases} 0 \text{ dB} & \omega \ll 10 \\ -3.01 \text{ dB} & \omega = 10 \\ -20 \log \omega & \omega \gg 10 \end{cases}$$



Phase:

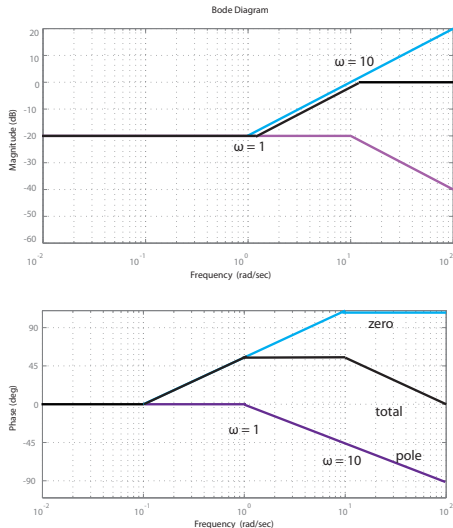
$$\angle \left(\frac{j\omega}{10} + 1 \right)^{-1} = \begin{cases} 0^\circ & \omega < 1 \\ -45^\circ \left(1 + \log \frac{\omega}{10} \right) & 1 < \omega < 100 \\ -90^\circ & \omega > 100 \end{cases}$$



Plotting Real Poles and Zeros

Lead-Lag

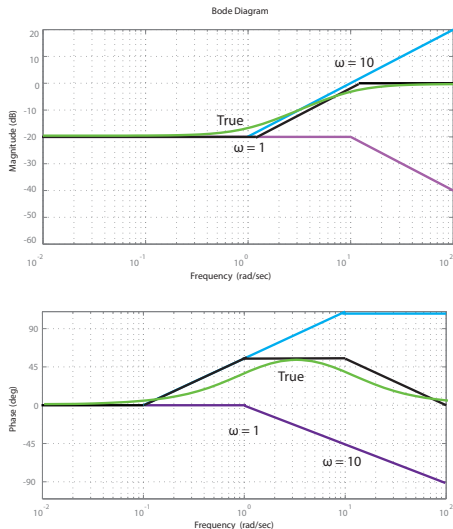
Add them all together



Plotting Real Poles and Zeros

Lead-Lag

Compare with the True Plot

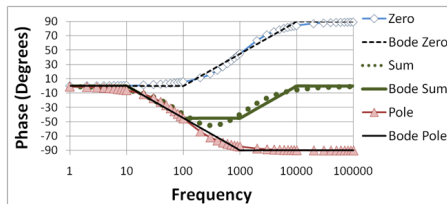
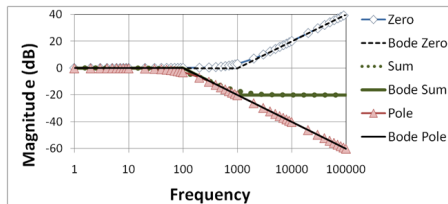


Plotting Real Poles and Zeros

Lead-Lag

Example: Lag Compensator

$$G(s) = \frac{1}{10} \frac{s + 1000}{s + 100} = \frac{\frac{s}{1000} + 1}{\frac{s}{100} + 1}$$



- Pole at $\omega = 100$
- Zero at $\omega = 1000$

Output Phase “Lags” behind the input.

Plotting Real Poles and Zeros

Lead-Lag

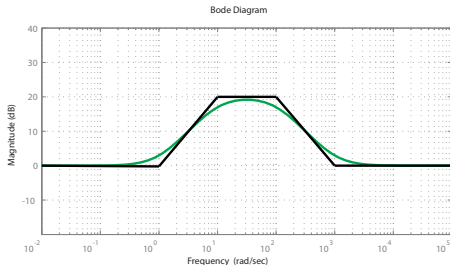
Put them together to get a **Lead-Lag Compensator**.

$$G(s) = \frac{s+1}{s+10} \frac{s+1000}{s+100} = \frac{s+1}{\frac{s}{10}+1} \frac{\frac{s}{1000}+1}{\frac{s}{100}+1}$$

- Zero at $\omega = 1$
- Pole at $\omega = 10$
- Pole at $\omega = 100$
- Zero at $\omega = 1000$

Magnitude: Changes in Slope.

- $\omega = 1$: +20
- $\omega = 10$: -20
- $\omega = 100$: -20
- $\omega = 1000$: +20



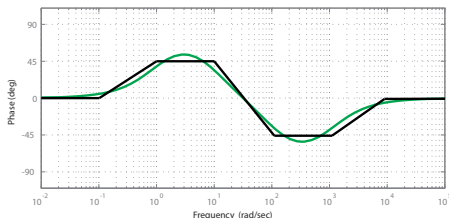
Plotting Real Poles and Zeros

Lead-Lag

$$G(s) = \frac{s+1}{s+10} \frac{s+1000}{s+100}$$

Phase: Changes in Slope.

- $\omega = .1$: $+45^\circ$
- $\omega = 1$: -45°
- $\omega = 10$: -90°
- $\omega = 100$: $+90^\circ$
- $\omega = 1000$: $+45^\circ$
- $\omega = 10000$: -45°



Complex Poles and Zeros

Now we move on to the final topic: **Complex Poles and Zeros**

Complex Zeros:

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

First Step: Put in standard form

$$G(s) = \omega_n^2 \left(\left(\frac{s}{\omega_n} \right)^2 + 2\zeta \frac{s}{\omega_n} + 1 \right)$$

- So $y_{ss} = 1$

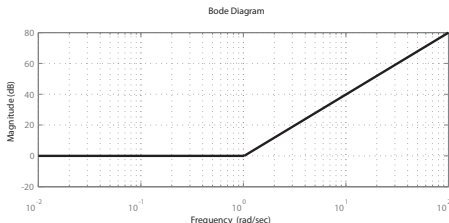
Ignoring the constant

$$G_1(j\omega) = - \left(\frac{\omega}{\omega_n} \right)^2 + 2\zeta \frac{\omega}{\omega_n} j + 1 = \begin{cases} 1 & \frac{\omega}{\omega_n} \ll 1 \\ 2\zeta j & \frac{\omega}{\omega_n} = 1 \\ - \left(\frac{\omega}{\omega_n} \right)^2 & \frac{\omega}{\omega_n} \gg 1 \end{cases}$$

Complex Poles and Zeros

Magnitude:

$$20 \log |G_1(j\omega)| = 20 \log \left| - \left(\frac{\omega}{\omega_n} \right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right|$$
$$= \begin{cases} 0 \text{ dB} & \frac{\omega}{\omega_n} \ll 1 \\ 20 \log(2\zeta) & \frac{\omega}{\omega_n} = 1 \\ 40(\log \omega - \log \omega_n) & \frac{\omega}{\omega_n} \gg 1 \end{cases}$$



Complex Poles and Zeros

The Behavior near $\omega = \omega_n$ depends on ζ .

- When $\zeta = 0$,

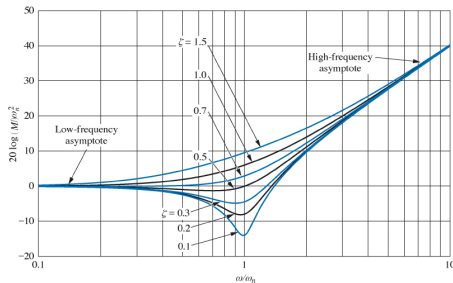
$$20 \log |G_1(i\omega)| = 20 \log(2\zeta) = -\infty$$

- When $\zeta = .5$,

$$20 \log |G_1(i\omega)| = 20 \log 1 = 0$$

- When $\zeta = 1$,

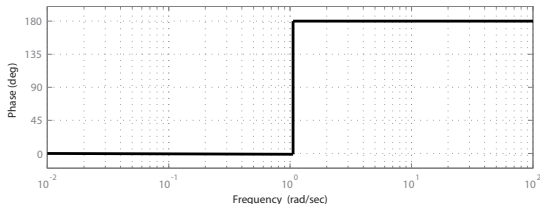
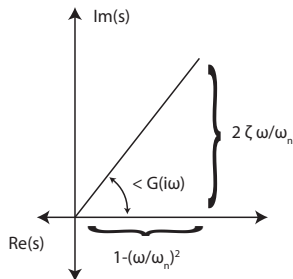
$$20 \log |G_1(i\omega)| = 20 \log 2 = 6.02dB$$



Complex Poles and Zeros

Phase:

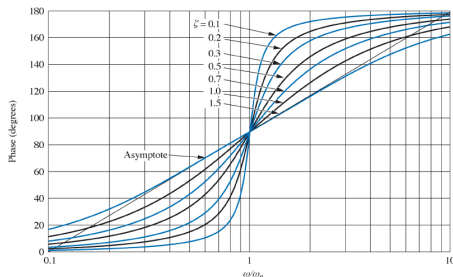
$$\begin{aligned}\angle G_1(i\omega) &= \angle \left(1 - \left(\frac{\omega}{\omega_n} \right)^2 + 2\zeta \frac{\omega}{\omega_n} i \right) \\ &= \tan^{-1} \left(\frac{2\zeta \omega \omega_n}{\omega_n^2 - \omega^2} \right) \\ &= \begin{cases} 0^\circ & \frac{\omega}{\omega_n} \ll 1 \\ 90^\circ & \frac{\omega}{\omega_n} = 1 \\ 180^\circ & \frac{\omega}{\omega_n} \gg 1 \end{cases}\end{aligned}$$



Complex Poles and Zeros

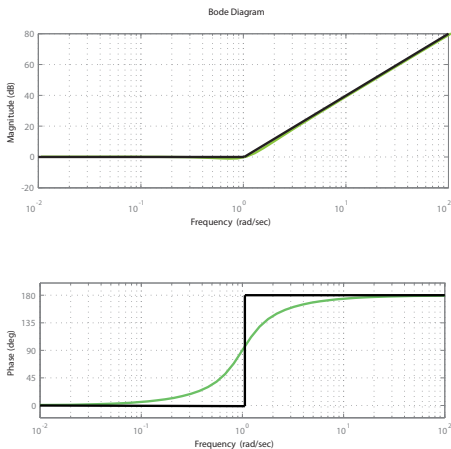
The Behavior near $\omega = \omega_n$ depends on ζ . For $\omega = \omega_n = 1$,

- When $\zeta = 0$, $\frac{d\angle G_1(j\omega)}{d\omega} = \infty$
- When $\zeta = .1$, $\frac{d\angle G_1(j\omega)}{d\omega} = 10$
- When $\zeta = .5$, $\frac{d\angle G_1(j\omega)}{d\omega} = 2$
- When $\zeta = 1$, $\frac{d\angle G_1(j\omega)}{d\omega} = 1$



Complex Poles and Zeros

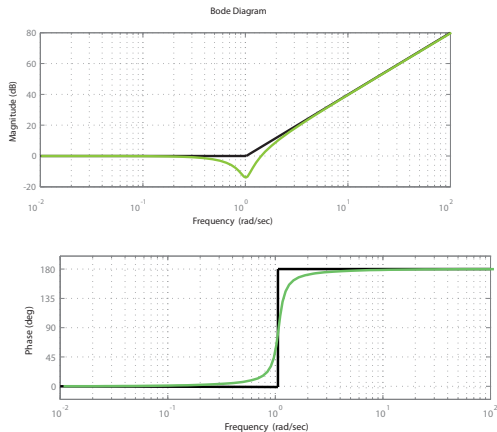
Comparison vs. True for $\omega_n = 1$, $\zeta = .5$.



Good for magnitude, bad for phase

Complex Poles and Zeros

Comparison vs. True for $\omega_n = 1$, $\zeta = .1$.



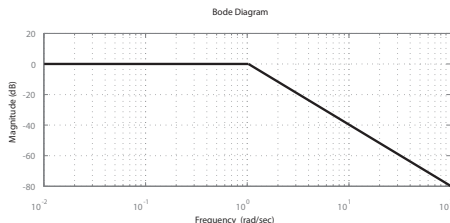
Complex Poles

Treatment of Complex Poles is similar
Complex Poles:

$$G(s) = \frac{1}{\left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right)}$$

Magnitude:

$$\begin{aligned} 20 \log |G_1(j\omega)| &= -20 \log \left| 1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} \right| \\ &= \begin{cases} 0dB & \frac{\omega}{\omega_n} \ll 1 \\ -20 \log(2\zeta) & \frac{\omega}{\omega_n} = 1 \\ -40(\log \omega - \log \omega_n) & \frac{\omega}{\omega_n} \gg 1 \end{cases} \end{aligned}$$



Complex Poles

The Behavior near $\omega = \omega_n$ depends on ζ .

- When $\zeta = 0$,

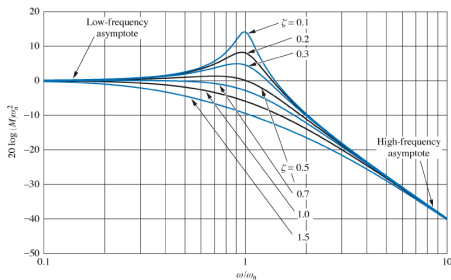
$$20 \log |G_1(j\omega)| = 20 \log(2\zeta) = +\infty$$

- When $\zeta = .5$,

$$20 \log |G_1(j\omega)| = 20 \log 1 = 0$$

- When $\zeta = 1$,

$$20 \log |G_1(j\omega)| = 20 \log 2 = -6.02dB$$



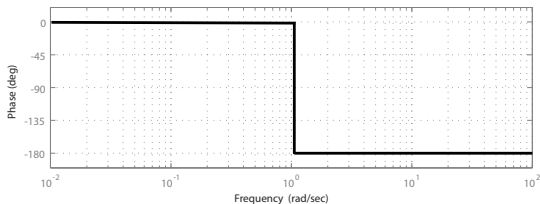
Complex Poles and Zeros

Figure: The Tacoma Narrows Bridge

Complex Poles

Phase:

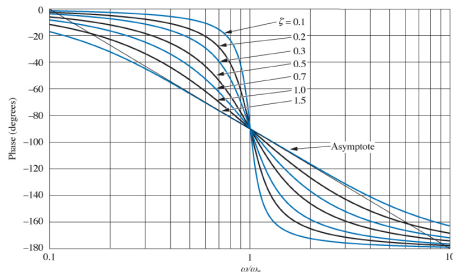
$$\begin{aligned}\angle G_1(j\omega) &= -\angle \left(1 - \left(\frac{\omega}{\omega_n} \right)^2 + 2\zeta \frac{\omega}{\omega_n} j \right) \\ &= \begin{cases} 0^\circ & \frac{\omega}{\omega_n} \ll 1 \\ -90^\circ & \frac{\omega}{\omega_n} = 1 \\ -180^\circ & \frac{\omega}{\omega_n} \gg 1 \end{cases}\end{aligned}$$



Complex Poles

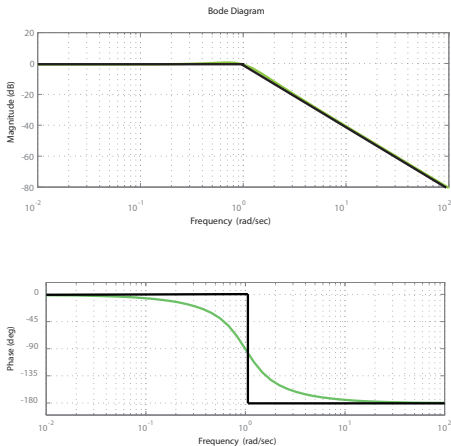
The Behavior near $\omega = \omega_n$ depends on ζ . For $\omega = \omega_n = 1$,

- When $\zeta = 0$, $\frac{d\angle G_1(j\omega)}{d\omega} = -\infty$
- When $\zeta = .1$, $\frac{d\angle G_1(j\omega)}{d\omega} = -10$
- When $\zeta = .5$, $\frac{d\angle G_1(j\omega)}{d\omega} = -2$
- When $\zeta = 1$, $\frac{d\angle G_1(j\omega)}{d\omega} = -1$



Complex Poles

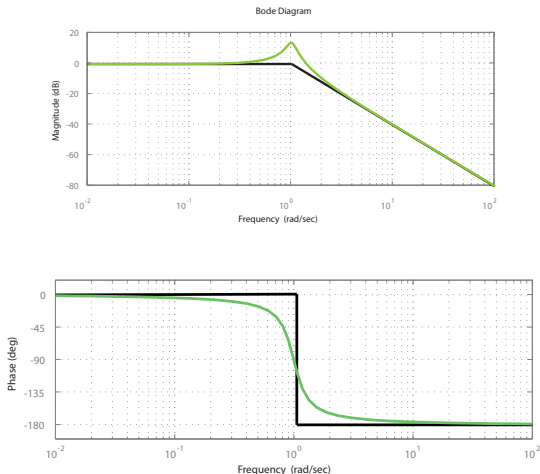
Comparison vs. True for $\omega_n = 1$, $\zeta = .5$.



Good for magnitude, bad for phase

Complex Poles

Comparison vs. True for $\omega_n = 1$, $\zeta = .1$.



Phase improves, Magnitude gets worse.

Summary

What have we learned today?

Simple Plots

- Real Zeros
- Real Poles
- Complex Zeros
- Complex Poles

Next Lecture: Compensation in the Frequency Domain