Spacecraft Dynamics and Control

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Lecture 2: Physical Invariants

Summary

In this Lecture, you will learn:

N-body Problem

- Introduction
- Invariants
  - Linear Momentum
  - Angular Momentum
  - Energy

Two-Body Problem

- How to calculate velocity given position
- How to calculate escape velocity

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Two-Body Problem
- How to calculate velocity given position
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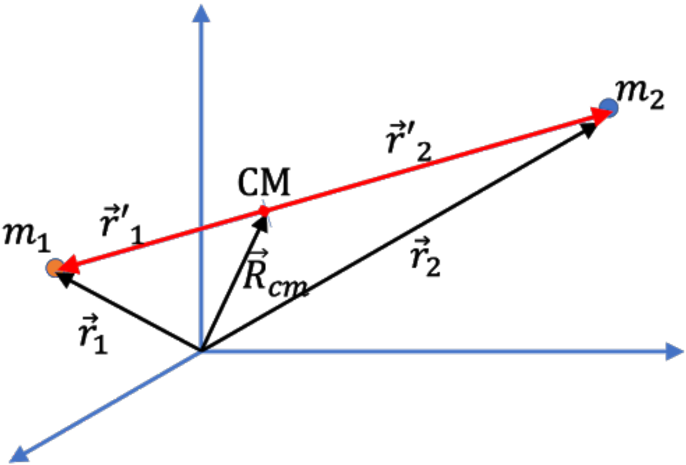
- This Lecture covers roughly Sections 1.2-1.4 of Prussing/Conway 2nd ed
- In this lecture, we will introduce the physical invariants of an orbit - Energy and angular momentum (also linear momentum, but this is not important).
- In the next lecture, we will translate these physical invariants to geometric invariants which describe the ellipse. These geometric invariants are called the orbital elements.
- Combined, these two lectures allow us to take a single observation of $\vec{r}$ and $\vec{v}$ and determine Energy and Angular momentum, which then allows us to calculate our orbital elements.

Universal Gravitation



$$\vec{F}_1 = G\frac{m_1 m_2}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

Relative motion (2-Body Motion)

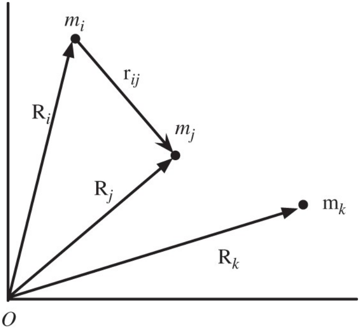
The force on mass 1 due to mass 2 is

$$m_1 \ddot{\vec{R}}_1 = \vec{F}_1 = G\frac{m_1 m_2}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

where we denote $\vec{r}_{12} = \vec{R}_2 - \vec{R}_1$. Clearly $\vec{r}_{12} = -\vec{r}_{21}$. The Force on mass 2 due to mass 1 is

$$m_2 \ddot{\vec{R}}_2 = \vec{F}_2 = G\frac{m_2 m_1}{\|\vec{r}_{21}\|^3}\vec{r}_{21}.$$

The problem is a nonlinear coupled ODE with 6 degrees of freedom (12 with velocities).



Solution: Consider relative motion (only $\vec{r}_{12}$)

$$\ddot{\vec{r}}_{12} = -\frac{G(m_1 + m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

Now only 6DOF.

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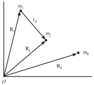

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Now only 6DOF.

- Put in first-order linear ODE form, there are 12 differential equations. The 12 states are $\vec{R}_1$ (3 states), $\vec{R}_2$ (3 states), $\dot{\vec{R}}_1$ (3 states), and $\dot{\vec{R}}_2$ (3 states). The nonlinearity comes from the $1/\|\vec{R}_1 - \vec{R}_2\|^3$ term.
- Simplifying the equations to relative motion ($\ddot{\vec{r}}_{12}$) reduces the number of states to 6, but this means knowledge of 6 independent states has been lost. The lost information is the position (3 states) and velocity (3 states) of the center of mass.
- G is the gravitational constant of the universe and is approximately $6.674 \cdot 10^{-11}\,\frac{m^3}{kg \cdot s^2}$.
- Source for image is P/C.

The N-body problem

In some situations, there are more than two bodies.

- The solar system
- The Earth-Moon system

In this case, the force on mass i due to all other masses is

$$m_i \ddot{\vec{R}}_i = \vec{F}_i = G \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij}$$

Definition 1.

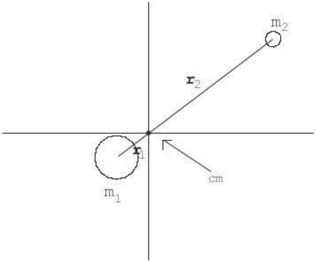
The center of mass of a collection of point masses m_i is

$$\vec{R}_{CM} = \frac{1}{\sum_i m_i} \sum_{i=1}^{N} m_i \vec{R}_i$$

The N-body problem

The key thing to note is that since $\vec{r}_{ij} = -\vec{r}_{ji}$,

$$\ddot{\vec{R}}_{CM} = \frac{1}{\sum_i m_i} \sum_{i=1}^{N} m_i \ddot{\vec{R}}_i$$
$$= \frac{G}{\sum_i m_i} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij} = 0$$



Therefore, The center of mass is *not accelerating*.

$$\vec{R}_{CM} = \sum_i m_i \vec{R}_i = \vec{C}_1 t + \vec{C}_2$$

$\vec{R}_{CM}$ makes an excellent choice for the origin of a **coordinate system**.

First Invariant Quantity: Linear Momentum

$$\vec{L} = \sum_i m_i \dot{\vec{R}}_i = \vec{C}_1$$

Thus the motion of the center of mass doesn't change with time.

- Technically, 3 invariants, since $\vec{L}$ is a vector.

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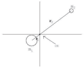

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- The “N-body problem” refers to the differential equations defined by N planets, all of which exert significant gravitational forces.
- The 3-Body problem as described in P/C, chapter 4 is a special case of the N-body problem when $N = 3$. However, we also use the term restricted 3-body problem to refer to the case where 2 bodies exert significant gravitational forces and one does not. This is the framework where we study stability of Libration points, as discussed in P/C in Chapter 4.2

Invariants in the N-body problem

We now define the two key invariant quantities which will define the motion.

- Angular Momentum
- Energy

These hold for the 2-, 3- and N- body problems. Begin with the angular momentum.

Definition 2.

The angular momentum of a collection of particles is

$$\vec{H}(t) = \sum_{i=1}^{N} m_i \vec{R}_i(t) \times \dot{\vec{R}}_i(t)$$

We will show next that

$$\begin{aligned}\frac{d}{dt}\vec{H}(t) &= \sum_{i=1}^{N} m_i \left(\dot{\vec{R}}_i(t) \times \dot{\vec{R}}_i(t) + \vec{R}_i(t) \times \ddot{\vec{R}}_i(t)\right) \\ &= \sum_{i=1}^{N} m_i \left(\vec{R}_i(t) \times \ddot{\vec{R}}_i(t)\right) = 0\end{aligned}$$

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- Recall that for vectors, $\vec{r} \times \vec{v} = \vec{n}\|\vec{r}\|\|\vec{v}\| \sin\theta$, where θ is the angle between the two vectors and $\vec{n}$ is a unit vector in the direction perpendicular to the plane generated by $\vec{r}$ and $\vec{v}$ and by the right-hand rule. Clearly $\vec{r} \times \vec{v} = -\vec{v} \times \vec{r}$.
- First Equation follows because the cross-product obeys the chain rule for differentiation, so
$$\frac{d}{dt}(r \times v) = \dot{r} \times v + r \times \dot{v}.$$
- The second equation is because $\vec{R} \times \vec{R} = 0$ for any vector, $\vec{R}$, since $\theta = 0$.

Conservation of Angular Momentum under Gravity

Begin with the relation

$$\ddot{\vec{R}}_i = \frac{\vec{F}_i}{m_i} = G \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij}.$$

Then

$$\sum_{i=1}^{N} m_i \left(\vec{R}_i \times \ddot{\vec{R}}_i \right) = G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{R}_i \times \vec{r}_{ij}.$$

However, we can use the identities

$$\vec{R}_i \times \vec{r}_{ij} = \vec{R}_i \times \left(\vec{R}_j - \vec{R}_i \right) = \vec{R}_i \times \vec{R}_j$$

$$\vec{R}_j \times \vec{R}_i = -\vec{R}_i \times \vec{R}_j$$

Thus

$$\sum_{i=1}^{N} m_i \left(\vec{R}_i \times \ddot{\vec{R}}_i \right) = G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{R}_i \times \vec{r}_{ij} = G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{R}_i \times \vec{R}_j = 0.$$

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- First equation comes from law of graviation, dividing by m_i.
- Second equation comes from the scalar multiplication and distributive properties of cross-products:

$$m(r \times v) = (mr) \times v = r \times (mv) \quad \text{and} \quad r \times (\sum_i v_i) = \sum_i (r \times v_i)$$

- First identity follows because $\vec{R} \times \vec{R} = 0$.
- In the last line, the first equality follows from the first identity. The second equality follows from the second identity.

$$\sum_i \sum_{\substack{j=1 \\ j \neq i}} R_i \times R_j = \sum_{i=1}^{N} \sum_{j=1}^{i-1} R_i \times R_j + \sum_{i=1}^{N} \sum_{j=i+1}^{N} R_i \times R_j$$

but

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} R_i \times R_j = \sum_{j=1}^{N} \sum_{i=1}^{j-1} R_i \times R_j = \sum_{i=1}^{N} \sum_{j=1}^{i-1} R_j \times R_i = \sum_{i=1}^{N} \sum_{j=1}^{i-1} -R_i \times R_j$$

More generally, this is a special case of the summation identity over the set X where X has the property that $(i, j) \in X$ implies $(j, i) \in X$:

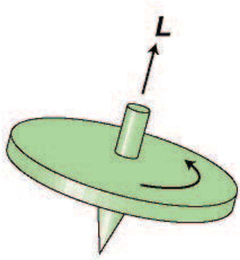
$$\sum_{(i,j) \in X} u_{ij} = \sum_{(j,i) \in X} u_{ji} = \sum_{(i,j) \in X} u_{ji}$$

Then $X := \{(i, j) \ : \ i \neq j\}$ and hence

$$\sum_{(i,j) \in X} R_i \times R_j = -\left(\sum_{(i,j) \in X} R_i \times R_j \right)$$

which implies $\sum_{(i,j) \in X} R_i \times R_j = 0$

Conservation of Angular Momentum



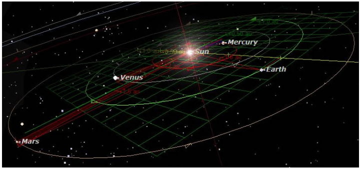


Figure: The Ecliptic Plane

Figure: Galactic Plane

Thus we conclude that

$$\dot{\vec{H}} = 0$$

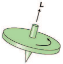
from whence we have that

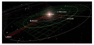
$$\vec{H} = \sum_{i=1}^{N} m_i \vec{R}_i \times \dot{\vec{R}}_i = C_3$$

The angular momentum defines a plane perpendicular to $\vec{H}$.

- For 2-bodies, this is the orbital plane
- For N-bodies, the ecliptic or galactic plane

Conservation of Angular Momentum

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- Galactic Plane is galaxy NGC 4452. Source: ESA/Hubble
- Ecliptic plane is from https://otherletter.com/another-science.html

Conservation of Angular Momentum when N is very large.

Milky Way + Andromeda

- Video is from Universe Sandbox and is a standard demo from that software
- In the video, each galaxy has a well-defined $\vec{H}_a$ and $\vec{H}_m$. After the collision, the new angular momentum of the combined galaxy is $\vec{H} = \vec{H}_a + \vec{H}_m$.
- Note that the combined galaxy for this purpose includes star systems which have escaped. Generally speaking, N-body systems are unstable in that planets or stars or systems may escape ($\lim_{t\to\infty} \vec{r}_{Nj}(t) = \infty$). If one does not account for the escaping mass, then conservation of both momentum and energy may be violated in this restricted sense.

Conservation of Energy

Definition 3.

The Kinetic Energy of a particle is

$$T_i = \frac{1}{2} m_i \dot{\vec{R}}_i^T \dot{\vec{R}}_i$$

Thus the total kinetic energy for a system of particles is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{R}}_i^T \dot{\vec{R}}_i$$

Definition 4.

The Gravitational Potential Energy of a collection of particles is defined to be

$$V = -\frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|}$$

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- Unlike momentum, the Kinetic Energy of the N-body system is NOT conserved, even though there is no external force. This is because some kinetic energy has been converted to potential energy.
- I emphasize the defined (for gravitational potential energy) since I propose and then prove that it is conserved. However, one could derive this quantity by starting with conservation of energy and working backward. In a sense, this is the approach taken by P/C in Section 1.2.
- GPE is negative because if it were positive, there would be no way to define a point of zero gravitational potential energy.

Conservation of Energy

We show that the fourth invariant is

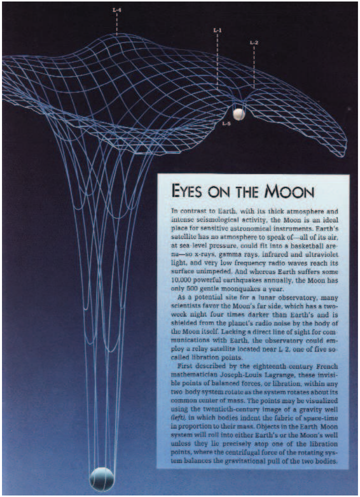
$$E = T + V = C_4$$

by showing $\dot{T} + \dot{V} = 0$.

$$\begin{aligned}\dot{T} &= \sum_{i=1}^{N} m_i \dot{\vec{R}}_i^T \ddot{\vec{R}}_i \\ &= G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \dot{\vec{R}}_i^T \vec{r}_{ij}\end{aligned}$$

Which is complicated.

However, now look at $\dot{V}$



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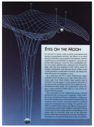

However, now look at $\dot{V}$

- The Illustration is of the potential energy of a third mass in the restricted earth-moon 3-body problem. It illustrates the libration point, which we will not discuss in depth in this class.
- Note that potential energy becomes infinitely negative near to the center of a point mass. This does imply that the kinetic energy would become infinitely large. However, we are saved from this non-physical mathematical inconvenience because the point-mass approximation of a planet fails once one crosses the mean surface of the planet. Moreover, for black holes, Newtonian physics break down at some point and so we would need to more rigorously examine the premise of conservation of energy here.

Conservation of Energy

Recall $\dot{T} = G \sum_{i=1}^{N} \sum_{\substack{j=1\\i\neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \dot{\vec{R}}_i^T \vec{r}_{ij}$ Now,

$$\dot{V} = -\frac{d}{dt}\frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|} = -\frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} m_i m_j \frac{d}{dt}(\vec{r}_{ij}^T \vec{r}_{ij})^{-\frac{1}{2}}$$

$$= \frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} m_i m_j (\vec{r}_{ij}^T \vec{r}_{ij})^{-\frac{3}{2}} \dot{\vec{r}}_{ij}^T \vec{r}_{ij}$$

$$= \frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} (\dot{\vec{R}}_j - \dot{\vec{R}}_i)^T \vec{r}_{ij} = -G \sum_{i=1}^{N} \sum_{\substack{j=1\\j\neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \dot{\vec{R}}_i^T \vec{r}_{ij}$$

Which cancels $\dot{T}$. Therefore

$$\dot{E} = \dot{T} + \dot{V} = 0$$

and hence gravity is a conservative field with $T(t) + V(t) = C_4$.

Conservation of Energy

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$$\begin{aligned}\dot{V} &= -\frac{d}{dt}\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|} = -\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j \frac{d}{dt}(\vec{r}_{ij}^T\vec{r}_{ij})^{-\frac{1}{2}}\\ &= \frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j (\vec{r}_{ij}^T\vec{r}_{ij})^{-\frac{3}{2}}\dot{\vec{r}}_{ij}^T\vec{r}_{ij}\\ &= \frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|^3}(\dot{\vec{R}}_j - \dot{\vec{R}}_i)^T\vec{r}_{ij} = -G\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|^3}\dot{\vec{R}}_i^T\vec{r}_{ij}\end{aligned}$$

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- Recall $\|v\| = (v^T v)^{\frac{1}{2}}$
- In the second equality, $d/dt(r^T r) = 2(\dot{r}^T r)$
- Recall $\vec{R}_{ij} = \vec{R}_j - \vec{R}_i$ (And not the other way around!).
- The last equality follows from

$$\sum_{(i,j)\in X}\dot{\vec{R}}_j\cdot\vec{r}_{ij} = -\sum_{(i,j)\in X}\dot{\vec{R}}_j\cdot\vec{r}_{ji} = -\sum_{(j,i)\in X}\dot{\vec{R}}_i\cdot\vec{r}_{ij} = -\sum_{(i,j)\in X}\dot{\vec{R}}_i\cdot\vec{r}_{ij}$$

Potential Energy

Illustration of Potential Wells in the solar system.

Each plane of the horizon corresponds to a velocity (Magnitude of kinetic energy).

Back to the Two-Body problem

Going back to the two-body problem.... Recall our equation of motion

$$\ddot{\vec{r}}_{12} = -\frac{G(m_1 + m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

but this considers only relative motion. How to recover absolute position? Use a coordinate system centered at the center of mass (barycenter) so that

$$\vec{R}_{CM} = \frac{\vec{R}_1 m_1 + \vec{R}_2 m_2}{m_1 + m_2} = 0$$

Then we can recover from $\vec{r}_{12}$

$$\vec{R}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}_{12}$$

and

$$\vec{R}_2 = \frac{m_1}{m_1 + m_2}\vec{r}_{12}$$

If m_1 is a planet and m_2 is a satellite, then $\frac{m_2}{m_1+m_2} \cong 0$ and $\frac{m_1}{m_1+m_2} \cong 1$ and so

$$\vec{R}_1 \cong 0 \qquad \text{and} \qquad \vec{R}_2 \cong \vec{r}_{12}.$$

Back to the Two-Body problem

Going back to the two-body problem.... Recall our equation of motion

$$\ddot{\vec{r}}_{12} = -\frac{G(m_1+m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

but this considers only relative motion. How to recover absolute position? Use a coordinate system centered at the center of mass (barycenter) so that

$$\vec{R}_{CM} = \frac{\vec{R}_1 m_1 + \vec{R}_2 m_2}{m_1+m_2} = 0$$

Then we can recover from $\vec{r}_{12}$

$$\vec{R}_1 = -\frac{m_2}{m_1+m_2}\vec{r}_{12}$$

and

$$\vec{R}_2 = \frac{m_1}{m_1+m_2}\vec{r}_{12}$$

If m_1 is a planet and m_2 is a satellite, then $\frac{m_2}{m_1+m_2} \cong 0$ and $\frac{m_1}{m_1+m_2} \cong 1$ and so

$$\vec{R}_1 \cong 0 \quad \text{and} \quad \vec{R}_2 \cong \vec{r}_{12}.$$

The equation of motion of mass 1 w/r to the center of mass is given by

$$\begin{aligned}
\ddot{\vec{R}}_2 &= \frac{m_1}{m_1+m_2}\ddot{\vec{r}}_{12} \\
&= -\frac{m_1}{m_1+m_2}\frac{G(m_1+m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12} = -\frac{Gm_1}{\|\vec{r}_{12}\|^3}\vec{r}_{12} \\
&= -\frac{Gm_1}{\|\vec{r}_{12}\|^3}\frac{m_1+m_2}{m_1}\vec{R}_2 = -\frac{G(m_1+m_2)}{\|\vec{r}_{12}\|^3}\vec{R}_2 \\
&= -\frac{G(m_1+m_2)}{\|\vec{R}_2\|^3}\frac{m_1^3}{(m_1+m_2)^3}\vec{R}_2 \\
&= -\frac{Gm_1^3}{(m_1+m_2)^2}\frac{\vec{R}_2}{\|\vec{R}_2\|^3} = -\mu\frac{m_1^2}{(m_1+m_2)^2}\frac{\vec{R}_2}{\|\vec{R}_2\|^3}
\end{aligned}$$

If the mass of $m_1 >> m_2$, we make the approximation

$$\frac{Gm_1^3}{(m_1+m_2)^2} \cong Gm_1 = \mu$$

In all future calculations, we can obtains absolute orbits about the CM by using this more accurate value of "μ".

The Orbital Parameter, μ

While we are considering **orbits**, we can make some simplifications. Our first simplification is to write

$$\mu = G(m_1 + m_2).$$

If $m_2 >> m_1$, then

$$\mu \cong Gm_2$$

- Each central body has its own μ
- The size of the orbit varies with μ
- Needed to convert orbital elements to $\vec{r}$ and $\vec{v}$.
- These values are tabulated in P/C, Appendix 1

The Sun

Mass = $1.989 \cdot 10^{30}$ kg

Radius = $6.9599 \cdot 10^{5}$ km

$\mu_{sun} = Gm_{sun} = 1.327 \cdot 10^{11}$ km^3/s^2

The Earth

Mass = $5.974 \cdot 10^{24}$ kg

Radius = $6.37812 \cdot 10^{3}$ km

$\mu_{earth} = Gm_{earth} = 3.986 \cdot 10^{5}$ km^3/s^2

Mean distance from sun = 1 au = $1.495978 \cdot 10^{8}$ km

The Moon

Mass = $7.3483 \cdot 10^{22}$ kg

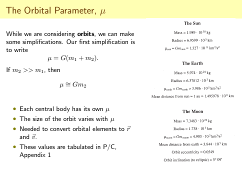
Radius = $1.738 \cdot 10^{3}$ km

$\mu_{moon} = Gm_{moon} = 4.903 \cdot 10^{3}$ km^3/s^2

Mean distance from earth = $3.844 \cdot 10^{5}$ km

Orbit eccentricity = 0.0549

Orbit inclination (to ecliptic) = $5° 09'$



Note that using the calculations on the previous slide, the magnitude of the motion of the earth due to the moon is

$$\frac{m_m}{m_m + m_e} r_{em} = 4652km$$

The radius of the earth is 6378km, so the earth-moon center of mass lies under the surface of the earth.

A note on the Gravitational Constant, G

The calculation of G is non-trivial

- Given G, it is easy to calculate the mass of any planet.
- The search for G was another major scientific quest.

In principle, it is easy to calculate:

- take two objects of known mass (m_1, m_2) and measure the attraction, F. Then

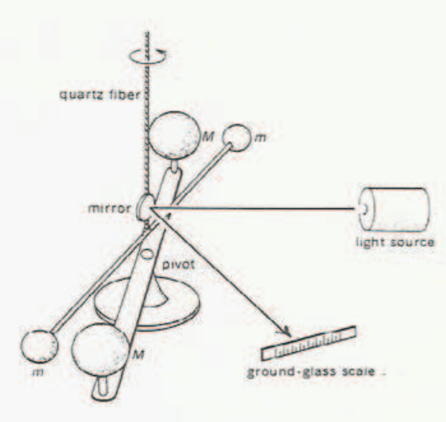
$$G = \frac{Fr^2}{m_1 m_2}$$

Unfortunately, the force is infinitesimal for all but planet-size objects. So how to calculate G?

The Cavendish Balance

The first accurate measurement of G was made by Cavendish in 1798.

1. Suspend two small spheres, separated by length, l, by a quartz filament.
2. Move two large spherical masses within known (small) distance of test masses.
3. Gravity will produce a moment on the quartz fiber, causing a deflection.
4. Deflection is measured by movement of a mirror on glass filament.
5. Rotation of mirror causes movement of reflected light.



The Cavendish Measurement revealed

$$G = 6.754 * 10^{-11} m^3 kg^{-1} s^{-2} (6.74(4))$$

A bit high, but OK for the time. ($6.67408 \pm .00031$ from CODATA)

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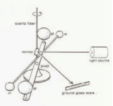

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- Newton guessed the graviational constant as $7 * 10^{-11} m^3 kg^{-1} s^{-2}$ which assumes the mean density of earth is 4 times as dense as water.
- Cavendish actually deduced density of the earth and not G, as use of the universal gravitational constant is a relatively modern notational convenience. There was a miscalculation in the published value, however,
- The force is strongest when r is small, so why use spheres, which are well-known to maximize distance?
- There is some disagreement over whether the gravitational constant varies with time. Modern measurements seem to return values which are inconsistent beyond expected deviation.
- One recent study (2014) measured $6.67191(99)10^{-11}$, another (2007) $6.693(34)10^{-11}$ (Fixler - unresolved as of 2018)
- Committee on Data for Science and Technology lists the official value at $6.67408 \pm .00031$ as of 2014. This is a reduction in uncertainty from 2010.
- illustration is not actually of the Cavendish experiment.

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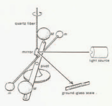

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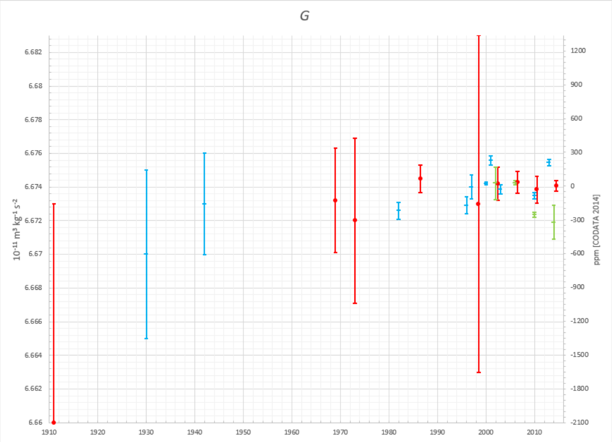
From Wikipedia: (torsion-balance measurements in blue; lit survey results in red; other experiments in green)

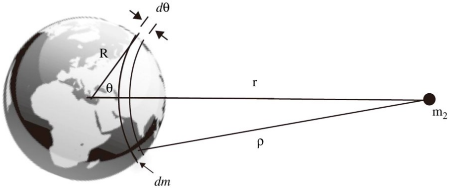


A Problem with the Point-Mass Model?

Non-Point Masses

- Our equation of motion assumes that the masses are concentrated at a point.
- Most masses are actually quite large (planet-sized)

Question: Is this a problem?
Answer: Not if there is symmetry about the line $\vec{r}_{12}$.



The *sphere* is symmetric about any line passing through the center.

- Most planets are spheres.
- Exception is the orbit perturbation effect due earth oblateness.
- See book for proof that the effect reduces to a point mass.

Now lets return to the question of potential fields and energy

For an orbit, we will now only consider motion of the satellite relative to the center of mass, which we denote

$$\vec{r}(t) \cong \vec{r}_{12}(t).$$

We will ignore motion of the larger body.

To begin, we introduce the non-dimensional versions of energy. This is energy per unit mass for the satellite.

Kinetic Energy:

$$T = \frac{1}{2}\|\vec{v}\|^2$$

Gravitational Potential Energy:

$$V = -\frac{\mu}{\|\vec{r}\|}$$

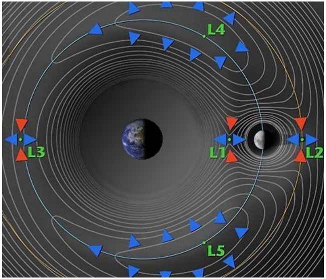
Conservation says that T+V is conserved. This can already be used to solve problems

Example: Velocity

Question: Suppose a satellite of earth is initially tracked at radius of $r_1 = 20,000km$ at a velocity of $1000m/s$. The satellite is later spotted at a radius of $10,000km$. Determine the velocity of the satellite.

Solution: Find the Energy at the initial time and use it to find the kinetic energy at the final time.

$$E = T_1 + V_1 = \frac{1}{2}\|\vec{v_1}\|^2 - \frac{\mu}{\|\vec{r_1}\|}$$
$$= .5 - \frac{398601}{20,000} = -19.43$$



$$V_2 = -\frac{\mu}{\|\vec{r_2}\|} = -\frac{398601}{10,000} = -39.86.$$

So $T_1 + V_1 = T_2 + V_2$ implies

$$T_2 = E - V_2 = -19.43 + 39.86 = 20.43$$
$$\|v_2\| = \sqrt{2T_2} = \sqrt{40.86} = 6.392km/s$$

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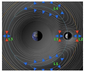

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- Don't forget the occasional conversion between altitude and radius. The radius of the earth is 6378 km, so any spacecraft launched on the surface of the earth begins with the corresponding amount of potential energy.
- Note that this use of the energy equation does not assume the object is in circular orbit. There is another equation which relates radius to velocity for objects in circular orbit, but we have not yet introduced it.
- This equation is typically used when we have two observations, but not enough information to determine all the orbital elements.

Example: Escape Velocity

Escape velocity is the kinetic energy needed to leave the sphere of influence of a planet. To achieve escape, net energy, E must be positive, so that as $r \to \infty$,

we still have forward motion.

At $r \to \infty$, $V_\infty = \lim_{r\to\infty} \frac{\mu}{\|\vec{r}\|} = 0$, so

$$E_\infty = T_\infty + V_\infty = T_\infty$$

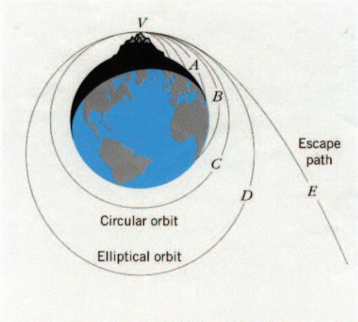
Question: Find the escape velocity at $r = 20,000km$.

Solution: As we know from the previous example, at $r = 20,000$, $V = -\frac{\mu}{r} = -19.93$. So in order for $E_1 = E_\infty > 0$, we need

$$E_1 = T_1 + V_1 = T_1 - 19.93 > 0$$

So we need $T > 19.93$. This yields a velocity of

$$\|v\| = \sqrt{2T_1} > 6.313$$



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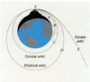

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- Don't forget if calculating the escape velocity at launch, to factor in a radius of 6378km.
- In contrast to the first example, we are using information on a second, desired observation to deduce information on a first, required radius and velocity.
- In this case, we never actually get to the second point, as it occurs at $t = \infty$. However, we assume its speed at this fictitious time is zero and its radius is $r = \infty$, so we can still use it as a valid observation.
- If $E > 0$, then we have reached escape velocity and are no longer in orbit. In this case, motion is hyperbolic, not elliptic. In this case, Kepler's laws don't apply.
- If $E < 0$, then we are still in elliptic orbit and Kepler's laws DO apply.

Conclusion

In this Lecture, you learned:

N-body Problem

- Introduction
- Invariants
  - Linear Momentum
  - Angular Momentum
  - Energy

Two-Body Problem

- How to calculate velocity given position
- How to calculate escape velocity

Next Lecture: The two-body problem continued

Derivation of Kepler's First Law

- eccentricity vector
  - How to calculate
  - circular orbits
  - elliptic orbits
  - parabolic orbits
  - hyperbolic orbits
- Solution to the two-body problem