Spacecraft Dynamics and Control

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Lecture 3: Planar Orbital Elements: True Anomaly, Eccentricity, and Semi-major Axis

Introduction

In this Lecture, you will learn:

Geometry of the orbit

- The polar equation.
- The planar orbital elements.
- A derivation of Kepler's 3 laws.
- The types of orbit
  - Elliptic
  - Parabolic
  - Hyperbolic

Different Representations

- How to convert between
  - Energy and Angular Momentum
  - geometric invariants (orbital elements)

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.

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In the first part of this lecture, we find one more semi-physical invariant - $\vec{e}$

- Equipped with $\vec{e}$, $\vec{h}$, and E, we find a *Solution* to the 2-body problem - The polar equation
- We then interpret the polar equation as an ellipse, parabola, or hyperbola
- We parameterize the ellipse using a and e, as well as p
- We show a 1-1 map between $E \rightleftarrows a$ and $h \rightleftarrows p$.

If we don't care about motion of the planet, the 2-body problem is actually a 1-body problem.

$$\ddot{\vec{r}}(t) = -\frac{\mu}{\|\vec{r}(t)\|^3}\vec{r}(t)$$

Last lecture, we showed that the Angular Momentum vector,

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

is invariant. i.e. $\dot{\vec{h}} = 0$.

Furthermore, the angular momentum vector is orthogonal to the orbital plane. That is

$$\vec{r}(t) \cdot \vec{h} = 0 \qquad \text{and} \qquad \vec{v}(t) \cdot \vec{h} = 0 \qquad \text{for all } t.$$

Difference Between Orbit and Flyby: Energy $E = T + V = \frac{1}{2}\|\vec{v}\|^2 - \frac{\mu}{\|\vec{r}\|}$

- If $E < 0$, object is in orbit (Elliptic Motion)
- If $E > 0$, motion is hyperbolic and does not repeat.

Question: How do we get from E and $\vec{h}$ to Kepler's laws?

Deriving Kepler's 3 laws

Recall Invariant Quantities: Energy and Angular Momentum

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- Unlike energy, angular momentum is a vector, which makes it more difficult to do simple calculations as we did with the energy equation.
- note that angular momentum is defined in unit mass coordinates.
- The next question we have is whether Energy and Angular momentum (4 invariants) are enough information to uniquely determine the orbit of a spacecraft or satellite.
- The answer is almost. The invariant Keplerian orbital elements are semi-major axis, eccentricity, inclination, Right Ascension of Ascending Node, and argument of periapse. All of these except argument of periapse can be determined from E and $\vec{h}$. Actually, since there are 5 invariant Keplerian elements, it is not surprising that we need additional information beyond the 4 measurements discussed. However, since inclination, RAAN, and argument of periapse are not introduced until the lecture on orientation of the orbit in space, this discussion can wait.
- For now, however, E and $\vec{h}$ are more than enough information to determine a and e which define the orbital ellipse. Indeed, we will show that the only the magnitude of h (reducing to 2 invariants) is needed to compute this ellipse.
- However, before simplifying, we will actually introduce 3 new invariants in the form of the eccentricity vector.

Definition 1.

The **Eccentricity Vector** can be defined as

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

Now we show that the eccentricity vector is invariant. Since we have already shown $\dot{\vec{h}} = 0$, we get

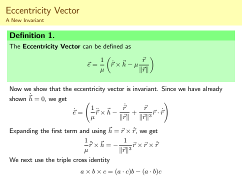
$$\dot{\vec{e}} = \left(\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} - \frac{\dot{\vec{r}}}{\|\vec{r}\|} + \frac{\vec{r}}{\|\vec{r}\|^3}\vec{r} \cdot \dot{\vec{r}}\right)$$

Expanding the first term and using $\vec{h} = \vec{r} \times \dot{\vec{r}}$, we get

$$\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} = -\frac{1}{\|\vec{r}\|^3}\vec{r} \times \vec{r} \times \dot{\vec{r}}$$

We next use the triple cross identity

$$a \times b \times c = (a \cdot c)b - (a \cdot b)c$$



Eccentricity Vector

A New Invariant

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- We are now working with dot products. These are slightly easier to work with than cross products.
- The first equation is an application of the chain rule to vectors which we also used in the previous lecture.

$$\frac{d}{dt}(\vec{r}^T\vec{r})^{-.5} = -.5(\vec{r}^T\vec{r})^{-1.5}\frac{d}{dt}(\vec{r}^T\vec{r}) = -(\vec{r}^T\vec{r})^{-1.5}\vec{r}^T\dot{\vec{r}} = -\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}$$

- With the eccentricity vector, we can also determine the argument of periapse. However, a discussion of argument of periapse will have to wait until a later lecture.
- It is important to note that the eccentricity vector cannot be derived from E and $\vec{h}$ alone. That is, these vectors are partially independent.
- However, the *magnitude* of $e = \|\vec{e}\|$ can be deduced from h and E.
- Originally proposed by W.R. Hamilton: inventor of quaternions and of "Hamiltonian" and "Cayley-Hamilton" fame (used in control theory).

The Eccentricity Vector is Invariant

Applying the triple cross identity,

$$\begin{aligned}\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} &= -\frac{1}{\|\vec{r}\|^3}\vec{r} \times \vec{r} \times \dot{\vec{r}} \\ &= -\frac{1}{\|\vec{r}\|^3}\left((\vec{r}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\vec{r})\dot{\vec{r}}\right) \\ &= -\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}\vec{r} + \frac{1}{\|\vec{r}\|}\dot{\vec{r}}\end{aligned}$$

Substituting,

$$\begin{aligned}\dot{\vec{e}} &= \left(\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} - \frac{\dot{\vec{r}}}{\|\vec{r}\|} + \frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right) \\ &= \left(-\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}\vec{r} + \frac{1}{\|\vec{r}\|}\dot{\vec{r}} - \frac{\dot{\vec{r}}}{\|\vec{r}\|} + \frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right) = 0\end{aligned}$$

Definition 2 (First Orbital Element).

The eccentricity of an orbit is

$$e = \|\vec{e}\|$$

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- For the third equality, we recall that the inner product $\vec{r} \cdot \vec{r} = \vec{r}^T\vec{r} = \|\vec{r}\|^2$.
- For now, we only need to know the magnitude of $\vec{e}$, which we denote as the scalar e (as $h = \|\vec{h}\|$).
- Later, we will discover that $\vec{e}$ is a vector which points from the center of the ellipse to the periapse (perigee).

The First Keplerian Element: Eccentricity

Now that we have a fixed vector in the orbital plane, lets measure position with respect to this vector.

- We will derive the polar equation of an ellipse:

$$r(t) = \frac{h^2/\mu}{1 + e\cos(f(t))} \qquad \cos f(t) = \frac{\vec{r}(t) \cdot \vec{e}}{r(t)e}$$

Assertions we will prove:

- The eccentricity *vector*, $\vec{e}$, points from the center or focus of the orbit towards the point of closest approach.
  - However, its *magnitude*, e, has a very different geometric Meaning.
- e is the classical elliptic parameter and measures the flattening of the ellipse.
- For a circular orbit, $e = 0$.
- For an elliptic orbit $(E < 0)$, $0 < e < 1$.
- For a hyperbolic orbit, $E > 0$, $e > 1$.

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- To prove these assertions, we will find an analytic expression for the position $r(t)$ as a function of time.

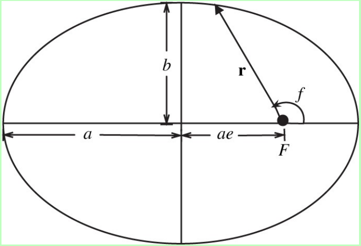
True Anomaly

Our first step is to replace time, t, with a more convenient and observable substitute variable, $f(t)$.

Definition 3 (Second Orbital Element).

The **True Anomaly**, $f(t)$ is the time-varying angle between $\vec{e}$ and $\vec{r}(t)$.

$$\cos f(t) = \frac{\vec{r}(t) \cdot \vec{e}}{\|\vec{r}(t)\| \|\vec{e}\|}$$



The true anomaly

- Varies between 0 and 2π
- Is easily measured from the focus point.
- However, we do not assert that $\dot{f}(t)$ is constant (unlike $\dot{t} = 1$).
- To make predictions, we will need to convert between t and $f(t)$.

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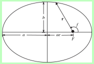

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- Converting between t and $f(t)$ is through intermediary mean and eccentric anomalies, which we will discuss later.

Solving the 2-Body problem as a function of h, e, and f

Recall:

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

Therefore, since $\vec{r} \cdot (\dot{\vec{r}} \times \vec{h}) = (\vec{r} \times \dot{\vec{r}}) \cdot \vec{h}$ and $\vec{r} \times \dot{\vec{r}} = \vec{h}$, we have

$$\begin{aligned} \mu\left(\vec{r} \cdot \vec{e}\right) &= \vec{r} \cdot \left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right) \\ &= \|\vec{h}\|^2 - \mu\|\vec{r}\| \end{aligned}$$

Now, by the definition of true anomaly ($\cos f(t) = \frac{\vec{r}(t) \cdot \vec{e}}{\|\vec{r}(t)\|\|\vec{e}\|}$), we get

$$\mu\vec{r}(t) \cdot \vec{e} = \mu\|\vec{r}(t)\|\|\vec{e}\| \cos f(t) = \|\vec{h}\|^2 - \mu\|\vec{r}(t)\|$$

Hence

$$\mu\|\vec{r}(t)\|(1 + \|\vec{e}\| \cos f(t)) = \|\vec{h}\|^2$$

From which we get the important equation

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

Is this a solution to the two-body problem?

Solution to the Two-Body Problem

Examine the equation

$$\|\vec{r}(t)\| = \frac{\|\vec{h}\|^2}{\mu\,(1 + \|\vec{e}\|\cos f(t))} = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

This equation gives the path on the orbital plane in polar coordinates (r, f) as f moves from $0\deg$ to $360\deg$ There are three cases

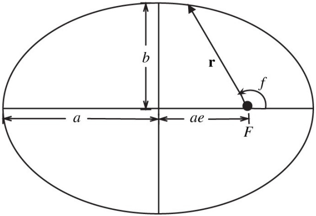


Figure: $\|\vec{e}\| < 1$: Elliptic

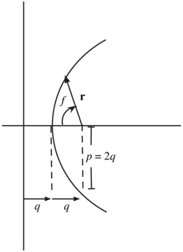


Figure: $\|\vec{e}\| = 1$: Parabolic

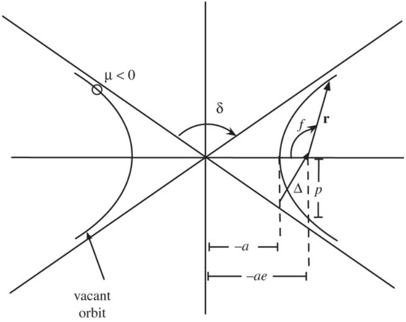
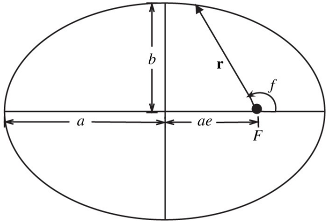


Figure: $\|\vec{e}\| > 1$: Hyperbolic

Consider these cases separately.

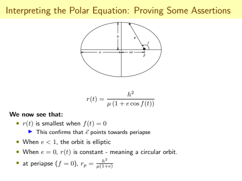
Interpreting the Polar Equation: Proving Some Assertions



$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

We now see that:

- $r(t)$ is smallest when $f(t) = 0$
  - This confirms that $\vec{e}$ points towards periapse
- When $e < 1$, the orbit is elliptic
- When $e = 0$, $r(t)$ is constant - meaning a circular orbit.
- at periapse ($f = 0$), $r_p = \frac{h^2}{\mu(1+e)}$



- To be complete, we should prove that $r = \frac{p}{1+e\cos\theta}$ is the polar representation of an ellipse centered at one of the foci. However, this it well-known and since it is not entirely trivial, most orbital mechanics texts do not include it. Recall the polar form of a set of points is the representation using angle and radius (r, θ) as opposed to (x, y) rectilinear coordinates.
- Now that we have shown that the motion is elliptic, the first remaining big question is how to use the invariant h and E to determine the Keplerian invariants a and e.
- We also need to prove Kepler's second and third laws.
- First we define the elements of an conic section (semi-latus rectum, semi-major axis for any conic) and semi-minor axis for an ellipse.

The Semi-Latus Rectum of an Ellipse/Hyperbola

Consider

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

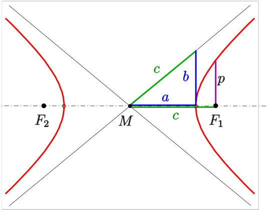
Point 1: At $f = \frac{\pi}{2}$:

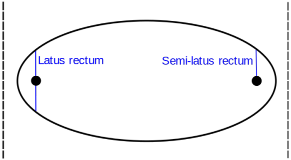
- $\vec{r}$ is orthogonal to $\vec{e}$. $\vec{r}$ points up from the major axis, parallel to the minor axis.
- This length is called the *semi-latus rectum*, denoted by p.

$$p = \frac{h^2}{\mu}$$

Although difficult to measure, p is a useful quantity

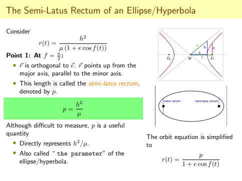
- Directly represents h^2/μ.
- Also called "`the parameter`" of the ellipse/hyperbola.





The orbit equation is simplified to

$$r(t) = \frac{p}{1 + e\cos f(t)}$$



- The semi-latus rectum, p, is defined in the same manner for ellipses, hyperbolae and parabolae. That is, it is not unique to the ellipse.
- Since μ is fixed, it is easy to convert back and forth between p and h. This is very useful in exam and homework problems when we can compute h or we would like to compute h given a and e.

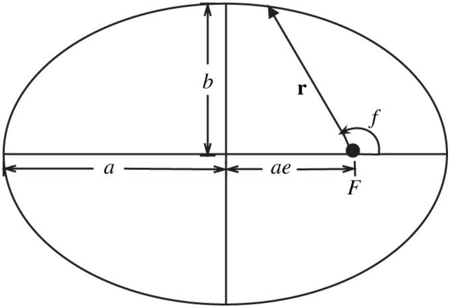
Periapse for all Orbits ($\|\vec{e}\| < 1$, $\|\vec{e}\| = 1$ or $\|\vec{e}\| > 1$)

At $f = 0$, $\vec{r}$ is aligned with $\vec{e}$ and the radius is *minimum*.

$$r_p = \frac{h^2}{\mu(1+e)} = \frac{p}{1+e}$$

The point of minimum radius is referred to as

- *Periapse* for a generic orbit.
- *Perigee* for orbits around Earth.
- *Perihelion* for orbits around the Sun.
- *Perilune* for lunar orbits.



The vector $\vec{e}$ always points toward the periapse. This orients the orbit in space. Note that $\|\vec{e}\| \neq r_p$.

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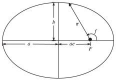

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Also,

- perihelion, (Our sun)
- perihermion, (Mercury)
- pericythe, (Venus)
- periareion, (Mars)
- perijove, (Jupiter)
- perichron (or perisaturnium) (Saturn),
- periuranion, (Uranus)
- periposeidion, (Neptune)
- perihadion, (Pluto)
- periastron, (Star)
- perigalacticon, (Galaxy)
- peribothron (or perimelasma) (Black hole)

The closest point of the moon in its orbit about earth is called “proxigee”

Apoapse for Elliptic Orbits ($\|\vec{e}\| < 1$)

At $f = \pi$**:**

- $\vec{r}$ is aligned with $-\vec{e}$.
- The radius r_a is now *maximum*

$$r_a = \frac{h^2}{\mu(1-e)} = \frac{p}{1-e}$$

The point of maximum radius is also referred to as

- *Apoapse* for generic orbits.
- *Apogee* for orbits around Earth.
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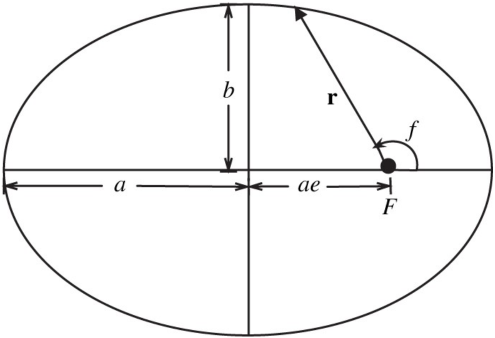
- *Apolune* for lunar orbits.



The Semi-Major Axis, a, for Ellipses and Hyperbolae

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

- For an ellipse, $a > 0$.
- For a hyperbola, $a < 0$.

Definition 4 (3rd Orb. Element).

The **semimajor axis** of the orbit is

$$a = \frac{p}{1 - e^2}$$

Alternatively, $p = a(1 - e^2)$!

Lemma 5.

The **semi-major axis** *is the radius of periapse divided by* $(1 - e)$*:*

$$a = \frac{p}{1 - e^2} = \frac{p}{1 + e}\frac{1}{1 - e} = \frac{r_p}{1 - e}$$

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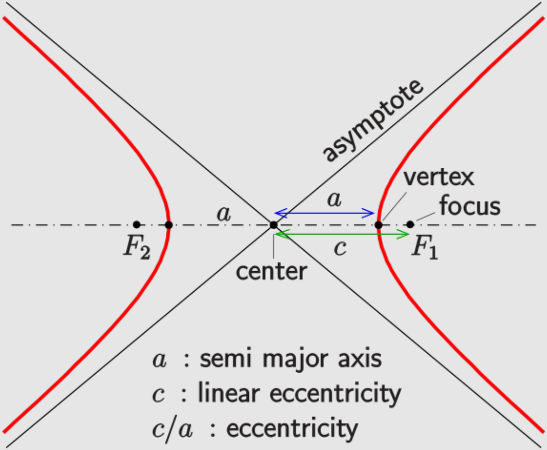
Alternatively, $r_p = a(1 - e)$!

- As $e \to 1$, $a \to \infty$. So semi-major axis is infinite for parabolae.
- In the plot below, $c = ae$



The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

For elliptic orbits, we can say a bit more!

For an **Ellipse, Apoapse** ($f = \pi$) is at

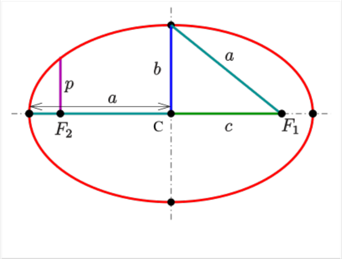
$$r_a = \frac{p}{1-e} = a(1+e)$$

Hence we have the relationship:

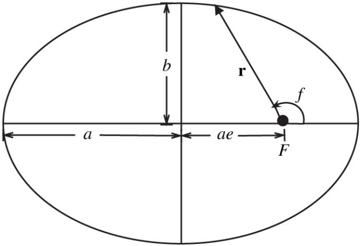
$$a = \frac{p}{1-e^2} = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{r_p + r_a}{2}$$

Summary: we can solve for p (and hence h) given

- r_a and e
- r_p and e
- a and e



$2a$ is the length of string used to draw the ellipse



The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

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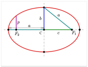

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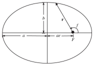

Summary of most commonly used formulae:

$$r_p = a(1-e)$$

$$r_a = a(1+e) \qquad \text{Ellipse Only!}$$

$$p = a(1-e^2)$$

$$a = \frac{r_p + r_a}{2} \qquad \text{Ellipse Only!}$$

$$p = \frac{h^2}{\mu}$$

Energy and Geometry: $E = -\frac{\mu}{2a}$ for Ellipses/Hyperbolae

Now, relate energy of the orbit directly to semi-major axis of the ellipse/hyperbola.

$$E = \frac{1}{2}v^2 - \frac{\mu}{r}$$

- We already know angular momentum in terms of geometry

$$p = \frac{h^2}{\mu}, \qquad \text{implies} \qquad h = \sqrt{p\mu}$$

Consider r and v at periapse. Then $r_p = a(1-e)$ and $r_p v_p = h$, so

$$E = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} = \frac{1}{2}\frac{p\mu}{a^2(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$= \frac{1}{2}\frac{\mu a(1-e^2)}{a^2(1-e)^2} - \frac{\mu}{a(1-e)} = \mu\frac{\frac{1}{2}(1+e)-1}{a(1-e)} = -\frac{\mu}{2a}$$

Therefore, the total energy of an orbit is

$$E = -\frac{\mu}{2a}$$

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Consider r and v at periapse. Then $r_p=a(1-e)$ and $r_pv_p=h$, so

$$\begin{aligned} E&=\frac{1}{2}v_p^2-\frac{\mu}{r_p} & &=\frac{1}{2}\frac{p\mu}{a^2(1-e)^2}-\frac{\mu}{a(1-e)}\\ &=\frac{1}{2}\frac{\mu a(1-e^2)}{a^2(1-e)^2}-\frac{\mu}{a(1-e)} & &=\mu\frac{\frac{1}{2}(1+e)-1}{a(1-e)}=-\frac{\mu}{2a} \end{aligned}$$

Therefore, the total energy of an orbit is

$$E=-\frac{\mu}{2a}$$

- The formula $E=-\frac{\mu}{2a}$ is incredibly useful.
- It applies to both hyperbolae and elliptic orbits.
- Given Energy, E and angular momentum, h, we can now obtain both a and e. a comes from energy. h gives $p=h^2/\mu$. Then solve $p=a(1-e^2)$ to get $e=\sqrt{1-\frac{p}{a}}$

$$E = -\frac{\mu}{2a}$$

Note for an elliptic orbit, the total energy is negative, as expected.

This yields the famous Vis-Viva relationship: $E = T + V$ or

$$-\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r} \qquad \text{which implies}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Also yields the "Excess Velocity":

$$v_{excess} = \lim_{r\to\infty} v(r) = \sqrt{-\frac{\mu}{a}}$$

Only real when $a \leq 0$.

Energy and Geometry

The Vis-Viva Equation for Elliptic/Hyperbolc Orbits

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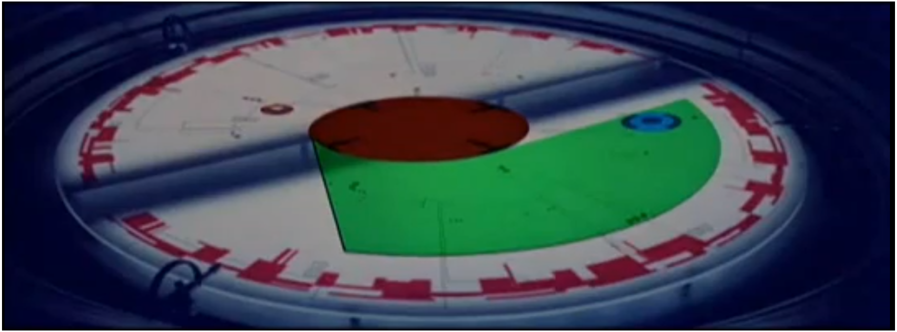
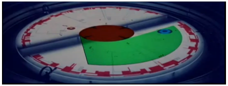
- Vis-Viva allows us to convert between v and r. A one-to-one relationship.
- Requires knowledge of a, but not e.
- Still applies to both ellipses and hyperbolae



For circular orbits,

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{r}\right) = \frac{\mu}{r} \qquad \text{or} \qquad v = \sqrt{\frac{\mu}{r}}$$

Circular Orbits
Vis-Viva Equation



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- In Star Wars 6, there is a tense moment as the death star is coming within range of the rebel base on the moon Yavin 4 which lies on the far side of the gas giant Yavin. The death star is in a lower circular orbit of Yavin than the moon and hence moves faster than the moon which is in a higher orbit.
- Tells us almost everything there is to know about circular orbits.

Example

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.
Solution: We first find the total energy, E from

$$E = -\frac{\mu}{2a}$$

where a can be found from

$$a = \frac{r_p + r_a}{2} = 20000km$$

Therefore

$$E = -\frac{\mu}{2a} = -9.96$$

Now we use

$$E = v^2/2 - \mu/r = -9.96$$

to find

$$v = \sqrt{2\left(\left(E + \frac{\mu}{r}\right)\right.} = \sqrt{2\left(-9.96 + 19.93\right)} = 4.464km/s$$

Elliptic Orbits: Verifying Kepler's 3 Laws

We have already established Kepler's First law: Elliptic Motion.

We have already established Kepler's First law: Elliptic Motion.

- Note that the smaller orbits are faster. Also within an orbit, velocities increase as one approaches the planet. However, while similar in effect, the cause of the latter is loss of potential energy. The cause of the former is the increase in velocity necessary to counteract increased gravity over shorter distances. This can be seen as a balance between gravitational acceleration and centripetal acceleration.

Kepler's Second Law

Equal Areas in Equal Time

Now its *Time* for Kepler's Second Law.

Now its *Time* for Kepler's Second Law.

- In the video, the grey and blue areas are of equal area. However, they clearly subtend a smaller angle (true anomaly) as measured from the planet.
- Recall that our position in the orbit is defined by true anomaly, f, which is the only orbital element which varies with time.
- It isn't really an orbital element, but rather a placeholder for time.
- However, $\dot{f} \neq constant$. So it is difficult for convert between $f(t)$ and t.
- Kepler's Second Law says that The rate of sweep of area is constant. $\dot{A} = constant$. This makes it easy to solve for ΔA.
- In the next lecture, we will talk about how to convert between A and f
- Today, we only prove that $\dot{A} = constant$.

Kepler's Second Law: Conservation of Angular Momentum

We need to find an expression for dA/dt.

Consider an infinitesimal swept section (triangle) of area:

$$dA \cong \frac{1}{2} r \cdot r \sin(df) \cong \frac{1}{2} r^2 df$$

where we denote $r = \|\vec{r}\|$. To first order,

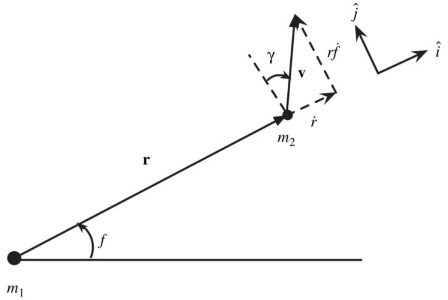
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{df}{dt}$$

- Note that $\dot{r} \neq \dot{\vec{r}}$. In the orbital plane,

$$\dot{\vec{r}} = \dot{r}\hat{i} + r\dot{f}\hat{j}$$

Now examine angular momentum

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^2\dot{f} \end{bmatrix}$$



Thus from conservation of angular momentum, we have

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{f} = \frac{\|\vec{h}\|}{2} = constant$$

Which proves Kepler's Second Law

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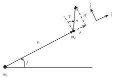

Thus from conservation of angular momentum, we have

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Which proves Kepler's Second Law

- An important consequence is that the rate of sweep of area is determined only by angular momentum, h.
- This means that the period of the orbit depends only on h.
- How to reconcile this with Kepler's third law??

The period of an orbit is the time taken to sweep out the entire ellipse.

Fortunately, the area of an ellipse is known

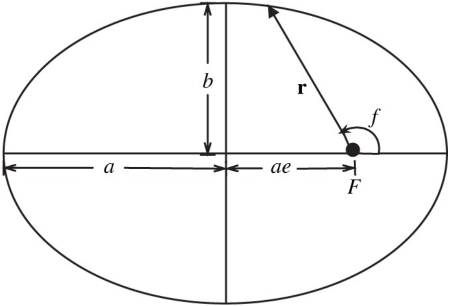
$$A_{ellipse} = \pi ab$$

where b is the **semi-minor axis** of the ellipse.

$$b = a\sqrt{1-e^2}$$

Therefore $dA = \frac{h}{2}dt$ implies

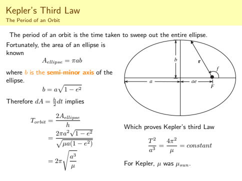
$$\begin{aligned} T_{orbit} &= \frac{2A_{ellipse}}{h} \\ &= \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}} \\ &= 2\pi\sqrt{\frac{a^3}{\mu}} \end{aligned}$$



Which proves Kepler's third Law

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mu} = constant$$

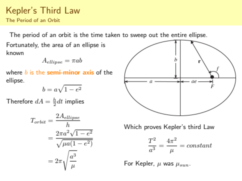
For Kepler, μ was μ_{sun}.



- The period of the orbit is the time it takes to sweep out $\pi ab = \pi a^2\sqrt{1-e^2}$ of area.
- The amount of area swept out in time ΔT is $\Delta A = \frac{h}{2}\Delta T$
- Recall $h = \sqrt{p\mu} = \sqrt{a(1-e^2)\mu}$
- The amount of time it takes to sweep out Area ΔA is

$$\Delta T = \frac{2}{h}\Delta A = 2\frac{\pi ab}{\sqrt{a(1-e^2)\mu}} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{a(1-e^2)\mu}} = 2\pi\sqrt{\frac{a^3}{\mu}}.$$

- The orbital period only depends on a, which only depends on Energy. e does not appear.
- The Equation has no meaning for hyperbolae.



A quick and dirty way of calculating Kepler's 3rd law for circular orbits:

- For an object in circular orbit, centripetal acceleration is in balance with gravitational acceleration, so that

$$\omega^2 r = \frac{Gm_p}{r^2} = \frac{\mu}{r^2}$$

- Now, if T is the period, $\omega = \frac{2\pi}{T}$ and hence

$$4\pi^2\frac{r}{T^2} = \frac{\mu}{r^2}$$

  or

$$\frac{r^3}{T^2} = \frac{\mu}{4\pi^2}$$

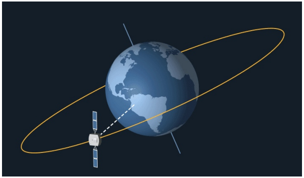
  or

$$\frac{T^2}{r^3} = \frac{4\pi^2}{\mu}$$

  which is Kepler's 3rd law!

- If you can calculate distance and period, you can calculate μ

Example: Geosynchronous Orbits



Problem: Suppose we want to position a satellite in equatorial orbit which always maintains the same position above the earth. Find a and e for such an orbit.

Solution: The earth rotates 366.25 times a year (the extra rotation compensates for the CCW motion of the earth about the sun). Thus the earth rotates once every 23 hours, 56 minutes and 4 seconds, implying the period of the satellite must be $\tau = 86164s$.

From Kepler's Third Law, we have

$$a = \left(\frac{\mu\tau^2}{4\pi^2}\right)^{1/3}$$
$$= 42,164km$$

Since the rotation rate of the earth is constant, we want $e = 0$.

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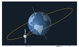

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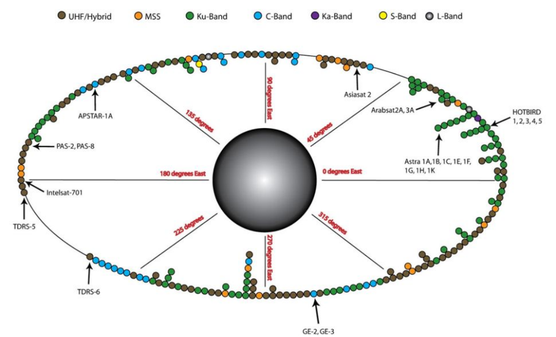
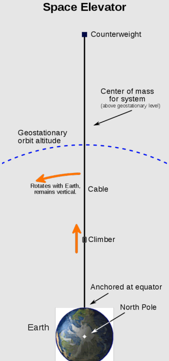
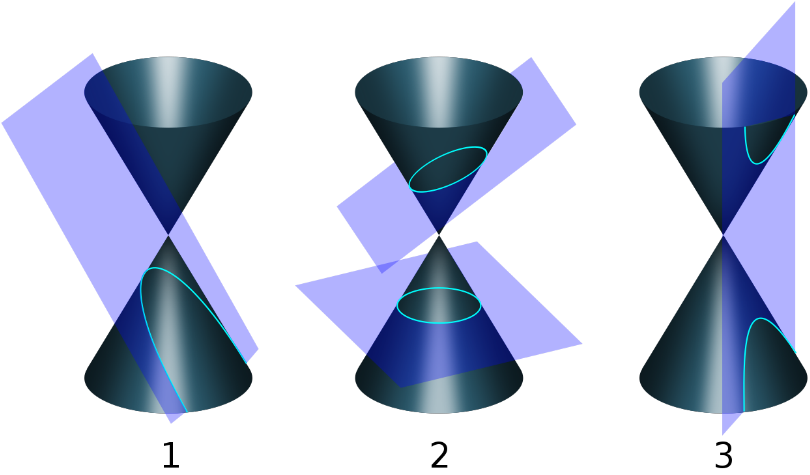
- There are 1800 geostationary slots
- There are 218 slots for North and South America (Phoenix is 112.0740 W)
- Availability is first-come first served
- Slots are assigned indefinitely and satellites may be replaced at end of life

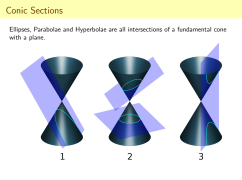




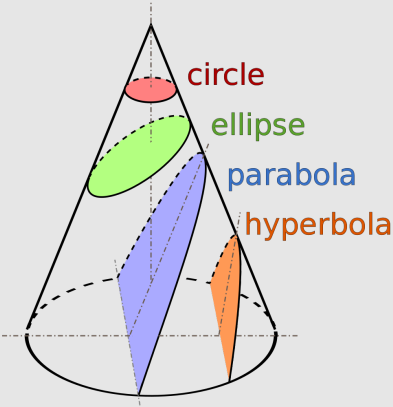
Conic Sections

Ellipses, Parabolae and Hyperbolae are all intersections of a fundamental cone with a plane.

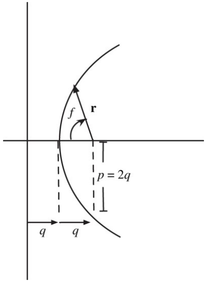


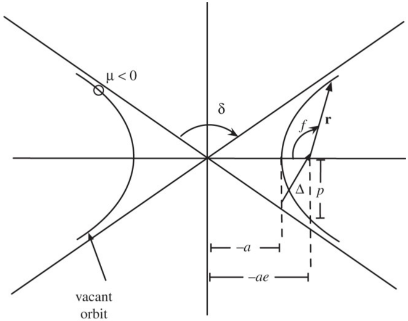


Conic sections are originally attributed to Menaechmus (d. 320BC). However the first rigorous treatment was by Apollonius of Perga (who also invented epicycles and deferents!)



$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$





Definition 6.

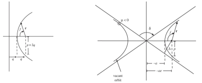
If $e = 1$, the orbit is **parabolic**.
$\delta = 180\,\mathrm{deg}$.

Definition 7.

If $e > 1$, the orbit is **hyperbolic**.
$\delta < 180\,\mathrm{deg}$

The case of Parabolic/Hyperbolic Orbits ($\|\vec{e}\| \geq 1$)

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$



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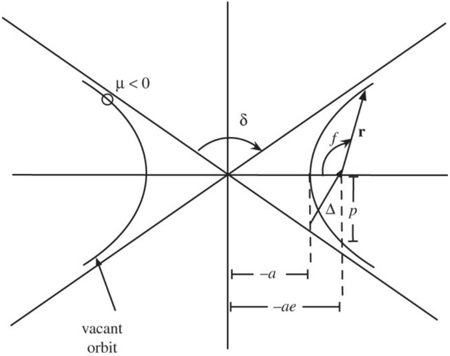
Definition 7.
If $e > 1$, the orbit is **hyperbolic**.
$\delta < 180\,\mathrm{deg}$

- δ is the turning angle. It measures the angular difference between the incoming trajectory and the outgoing trajectory.

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

If $e > 1$, then $\lim_t r(t) \to \infty$ when $1 + e\cos f = 0$.

- The orbit is asymptotic at $f = \pm\cos^{-1}\frac{1}{e}$.
    - $f = -\cos^{-1}\frac{-1}{e}$ is the incoming asymptote.
    - $f = \cos^{-1}\frac{-1}{e}$ is the outgoing asymptote.
- The angle between incoming and outgoing asymptotes is the *Turning Angle*, δ.
    - $\delta = 180^\circ - 2\cos^{-1}\frac{-1}{e} = 2\sin^{-1}\frac{1}{e}$



When $f \geq \delta/2$ or $f \leq -\delta/2$, the orbit is fictional.

- Hyperbolic orbits do not repeat.

Hyperbolic Orbits ($\|\vec{e}\| > 1$)

Turning Angle (δ)

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$

If $e > 1$, then $\lim_t r(t) \to \infty$ when $1 + e\cos f = 0$.

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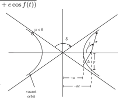
  - $\delta = 180^\circ - 2\cos^{-1}\frac{-1}{e} = 2\sin^{-1}\frac{1}{e}$



When $f \geq \delta/2$ or $f \leq -\delta/2$, the orbit is fictional.

- Hyperbolic orbits do not repeat.

- The turning angle parameter is unique to hyperbolae and parabolae.

Parabolic Orbits ($\|\vec{e}\| = 1$)

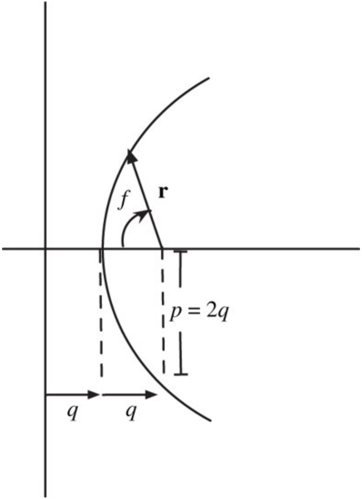
$$r(t) = \frac{h^2}{\mu\left(1 + \cos f(t)\right)}$$

Turning Angles: ($f \to \pm\pi$): If $e = 1$, then $\lim_t r(t) \to \infty$ as $\lim_t f(t) \to \pm\pi$.

- When $f = 0$, we have closest approach.

$$r_p = q = \frac{h^2}{2\mu} = \frac{p}{2}$$

- When $f \geq \pi$ or $f \leq -\pi$, the orbit is fictional.
  - Parabolic orbits do not repeat.
- The turning angle is $180\,\mathrm{deg}$.
- No "excess velocity"



Summary

Some Important Scalar Relations

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

$$r_p = a(1-e)$$

$$r_a = a(1+e)$$

$$p = a(1-e^2) = \frac{h^2}{\mu}$$

$$b = a\sqrt{1-e^2}$$

$$E = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$h = r_p v_p = r_a v_a \cong r^2 \dot{f}$$

Summary

Some Important Vector Relations

$$\begin{aligned}
\dot{\vec{e}} &= 0 \\
\dot{\vec{h}} &= 0 \\
\vec{h} &= \vec{r} \times \dot{\vec{r}} \\
\vec{e} &= \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right) \\
\vec{r}(t) \times \vec{h} &= 0 \\
\vec{e} \times \vec{h} &= 0 \\
\dot{\vec{r}} &= \dot{r}\hat{i} + r\dot{f}\hat{j}
\end{aligned}$$