Spacecraft Dynamics and Control

Matthew M. Peet

Lecture 5: Hyperbolic "Orbits" in Time

Introduction

In this Lecture, you will learn:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Most Equations for Elliptic Orbits also apply to Hyperbolic Trajectories

Properties of Keplerian Orbits

Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis $a < 0$ b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 + b^2/a^2} \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a\ (1 - e)$	$q = p/2$	$q = a\ (1 - e)$
Velocity, V, at distance, r, from Focus	$V^2 = \mu/r$	$V^2 = \mu\ (2/r - 1/a)$	$V^2 = 2\mu/r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, $\mathcal{E}$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = 0$	$\mathcal{E} = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu}$	$n = \sqrt{\mu/(-a)^3}$
Period, P	$P = 2\pi/n$	$P = 2\pi/n$	$P = \infty$	$P = \infty$
Anomaly	$v = M = E$	Eccentric anomaly, E $\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{v}{2} = D/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2}\tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e\sin E$	$M = qD + (D^3/6)$	$M = (e\sinh F) - F$
Distance from Focus, $r = q\,(1+e)/(1+e\cos v)$	$r = a$	$r = a\ (1 - e\cos E)$	$r = q + (D^2/2)$	$r = a\ (1 - e\cosh F)$
$r\,dr/dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{dv}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$	$\frac{dA}{dt} = \sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$

$\mu = GM$ is the gravitational constant of the central body; v is the true anomaly, and $M \equiv n\,(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion.

Given t, find r and v

For elliptic orbits:

1. Given time, t, solve for Mean Anomaly

$$M(t) = nt$$

2. Given Mean Anomaly, solve for Eccentric Anomaly

$$M(t) = E(t) - e \sin E(t)$$

3. Given eccentric anomaly, solve for true anomaly

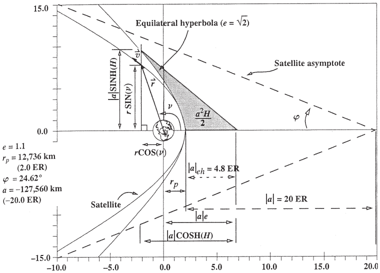
$$\tan \frac{f(t)}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E(t)}{2}$$

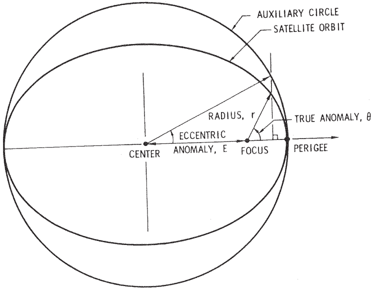
4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1 + e \cos f(t)}, \qquad v(t) = \sqrt{\mu \left(\frac{2}{r(t)} - \frac{1}{a} \right)}$$

Does this work for Hyperbolic Orbits? Lets recall the angles.

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}, \qquad v(t) = \sqrt{\mu\left(\frac{2}{r(t)} - \frac{1}{a}\right)}$$





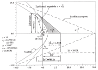
- True Anomaly is still the angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$, measured COUNTERCLOCKWISE.
- The polar equation still holds
- The Vis-viva equation still holds.

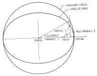
Conclusion: Step 4 Requires NO modifications.

Step 4:

True Anomaly and the Polar Equation

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}, \qquad v(t) = \sqrt{\mu\left(\frac{2}{r(t)} - \frac{1}{a}\right)}$$



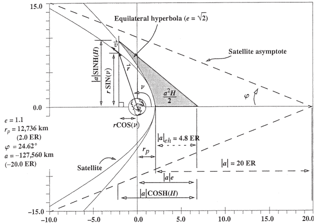


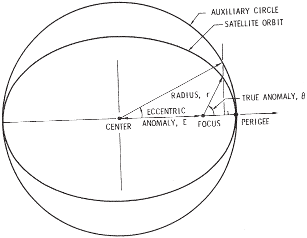
- True Anomaly is still the angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$, measured COUNTERCLOCKWISE.
- The polar equation still holds
- The Vis-viva equation still holds.

Conclusion: Step 4 Requires NO modifications.

- In the figure, θ is used for true anomaly. We typically use f. Occasionally, ν is also used in the texts.
- True anomaly is always well-defined for hyperbolic orbits.

$$M(t) = nt, \qquad M(t) = E(t) - e\sin E(t) \qquad \tan\frac{f(t)}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E(t)}{2}$$



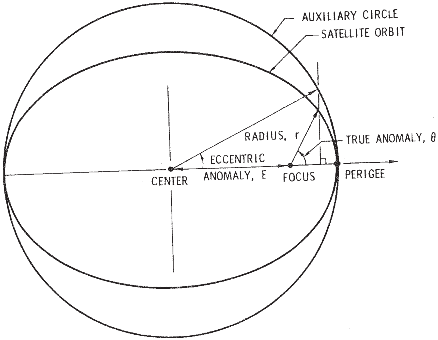


- Eccentric anomaly (E(t)) is measured from center of ellipse to a auxiliary reference circle.
  - For hyperbolic orbits, we use hyperbolic anomaly, based on a reference hyperbola.
- Mean Anomaly (M(t)) is the fraction of area of the ellipse which has been swept out, in radians.
  - Mean anomaly is not defined for hyperbolic orbits, as these orbits do not have a period.

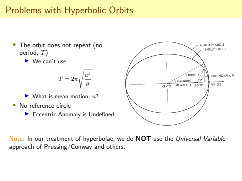
Conclusion: Steps 1,2, and 3 all need to be revisited.

Problems with Hyperbolic Orbits

- The orbit does not repeat (no period, T)
  - We can't use
    $$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$
  - What is mean motion, n?
- No reference circle
  - Eccentric Anomaly is Undefined



Note: In our treatment of hyperbolae, we do **NOT** use the *Universal Variable* approach of Prussing/Conway and others.

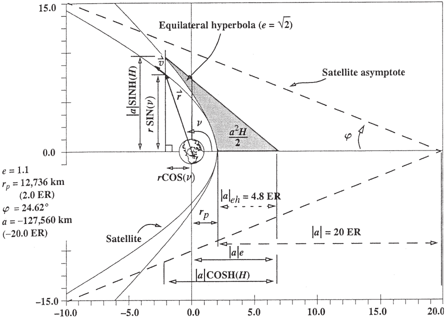


- The universal variable approach redefines the Kepler equation to be valid for both eccentric and hyperbolic orbits.
- Does not require us to know what type of orbit we have apriori.
- Useful for computer algorithms as it avoids case logic. Occasionally, students try and use Kepler's equation to solve hyperbolic orbit problems.
- No useful geometric interpretation, however.

Solutions for Hyperbolic Orbits: Step 3

Reference Hyperbola

Hyperbolic Anomaly is defined by the projection onto a reference hyperbola.



- defined using the reference hyperbola, tangent at perigee. Equation for reference hyperbola:

$$x^2 - y^2 = a^2$$

Hyperbolic anomaly (H) is the hyperbolic angle using the area enclosed by the center of the hyperbola, the point of perifocus and the point on the reference hyperbola directly above the position vector.

Solutions for Hyperbolic Orbits: Step 3
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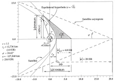

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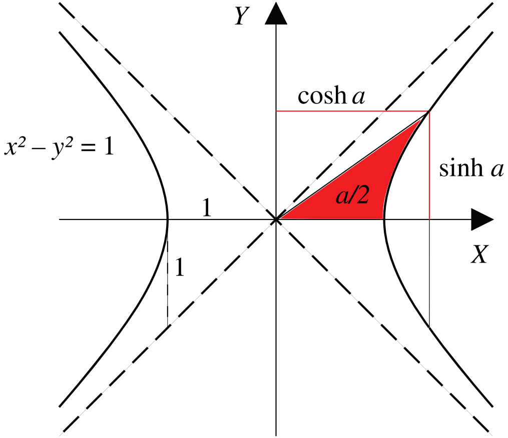
Hyperbolic anomaly (H) is the hyperbolic angle using the area enclosed by the center of the hyperbola, the point of perifocus and the point on the reference hyperbola directly above the position vector.

- The reference hyperbola is the hyperbola with an eccentricity of $\sqrt{2}$ whose periapse is the same as the periapse of the actual orbit.

Recall your Hyperbolic Trig.

Cosh and Sinh

Consider the Reference Hyperbola: $x^2 - y^2 = 1$



$cosh$ and $sinh$ relate area swept out by the reference hyperbola to lengths.

- Yet another branch of mathematics developed for solving orbits (Lambert).

Recall your Hyperbolic Trig.
Cosh and Sinh

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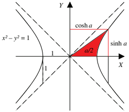

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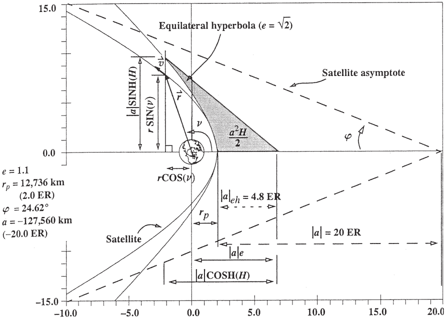
- Yet another branch of mathematics developed for solving orbits (Lambert).

- Recall that Kepler's 2nd law also applies to hyperbolic orbits (Equal Areas in Equal time ... $\dot{A} = h/2$)
- Defined using the normalized reference hyperbola.
- Lambert invented hyperbolic functions in the 18th century to compute the area of a hyperbolic triangle. We will meet Lambert again in a later lecture.
- See `en.wikipedia.org/wiki/Hyperbolic_function` for a thorough treatment of hyperbolic functions

Step 3: Hyperbolic Anomaly

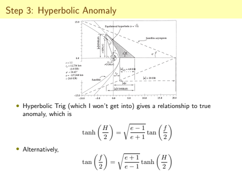


- Hyperbolic Trig (which I won't get into) gives a relationship to true anomaly, which is

$$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f}{2}\right)$$

- Alternatively,

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

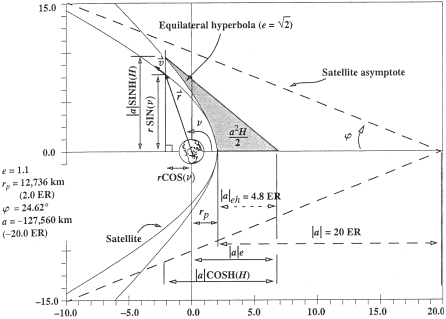


- Compare to the formulae for Eccentric anomaly.

$$\tan\frac{E(t)}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f(t)}{2}$$

$$\tan\frac{f(t)}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E(t)}{2}$$

Hyperbolic Anomaly



Using hyperbolic anomaly, we can give a simpler form of the polar equation.

$$r(t) = a(1 - e\cosh H(t))$$

Of course the original polar equation is still valid:

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

$$M(t) = nt \qquad\qquad n = \sqrt{\frac{\mu}{-a^3}}$$

Definition 1 (Hyperbolic Kepler's Equation).

$$M(t) = \sqrt{\frac{\mu}{-a^3}}t = e\sinh(H) - H$$

If we want to solve this for H, we get a different Newton iteration.

Newton Iteration for Hyperbolic Anomaly:

$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

with starting guess $H_1 = M$.

Hyperbolic Kepler's Equation: Steps 1 and 2

Mean Hyperbolic Anomaly (M(t)) and Mean Hyperbolic Motions (n)

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

$$M(t) = nt \qquad n = \sqrt{\frac{\mu}{-a^3}}$$

Definition 1 (Hyperbolic Kepler's Equation).

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Newton Iteration for Hyperbolic Anomaly:

$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

with starting guess $H_1 = M$.

- Compare to Kepler's equation and standard mean motion. But don't confuse them!

$$M(t) = E(t) - e\sin E(t)$$

$$M(t) = nt \qquad n = \sqrt{\frac{\mu}{a^3}}$$

Relation between M and f for Hyperbolic and Elliptic Orbits

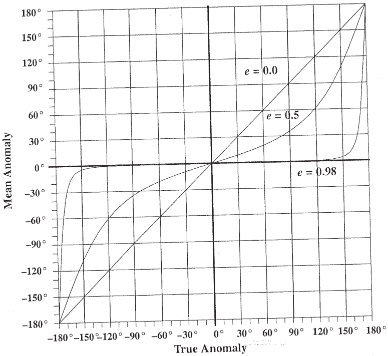


Figure: Elliptic Mean Anomaly vs. True Anomaly

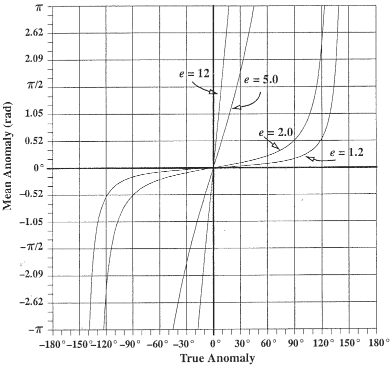
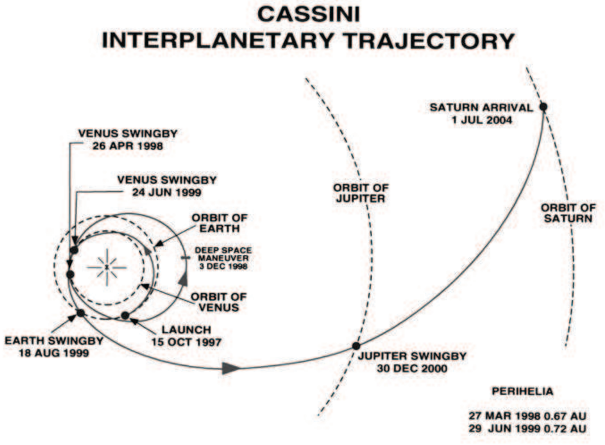


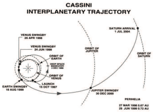
Figure: Hyperbolic Mean Anomaly vs. True Anomaly

Example: Jupiter Flyby



Problem: Suppose we want to make a flyby of Jupiter. The relative velocity at approach is $v_\infty = 10km/s$. To achieve the proper turning angle, we need an eccentricity of $e = 1.07$. Radiation limits our time within radius $r = 100,000km$ to 1 hour (radius of Jupiter is $71,000km$). Will the spacecraft survive the flyby?

Example: Jupiter Flyby



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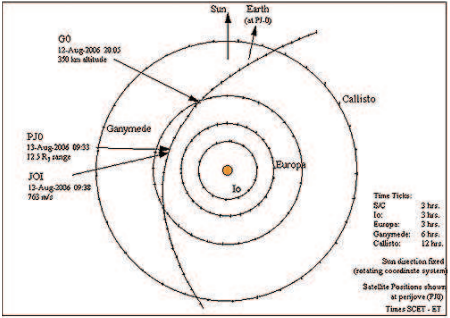
- The radiation in Jupiter's belts is said to be a million times greater than that in the Van Allen Belt of Earth.
- Pioneer 10 (at $r_p = 200,000km$) experienced 250,000 rads (500 rads is fatal to humans).
- This belt extends out to Europa, Io, Ganymede, and even Callisto - making human surface colonization of these moons problematic.
- Only on Callisto is human surface exploration considered feasible.
- Studying the Magnetosphere which produces this radiation is the primary goal of the Juno spacecraft (arrived in 2016 - perijove is $r_p = R_j + 4200km$).

Solution: First solve for a and p. $\mu = 1.267 \cdot 10^8$.



- The total energy of the orbit is given by

$$E_{tot} = \frac{1}{2}v_\infty^2$$

- The total energy is expressed as

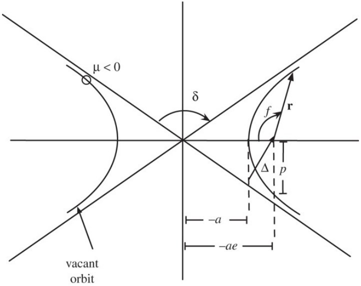
$$E = -\frac{\mu}{2a} = \frac{1}{2}v_\infty^2$$

which yields

$$a = -\frac{\mu}{v_\infty^2} = -1.267 \cdot 10^6$$

- The parameter is $p = a(1 - e^2) = 1.8359 \cdot 10^5$

Example Continued



We need to find the time between $r_1 = 100,000km$ and $r_2 = 100,000km$. Find f at each of these points.

- Start with the polar equation:

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

- Solving for f,

$$f_{1,2} = \cos^{-1}\left(\frac{1}{e} - \frac{r}{ep}\right) = \pm 64.8\,\mathrm{deg}$$

Example Continued

Given the true anomalies, $f_{1,2}$, we want to find the associated times, $t_{1,2}$.

- Only solve for t_2, get t_1 by symmetry.
- First find Hyperbolic Anomaly,

$$H_2 = 2\tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f_2}{2}\right)\right) = .2345$$

- Now use Hyperbolic anomaly to find mean anomaly

$$M_2 = e\sinh(H_2) - H_2 = 0.0187$$

  - This is the "easy" direction.
  - No Newton iteration required.
- t_2 is now easy to find

$$t_2 = M_2\sqrt{\frac{-a^3}{\mu}} = 2372.5s$$

Finally, we conclude $\Delta t = 2 * t_2 = 4745.1s = 79min$.

So the spacecraft may not survive.

The Method for Hyperbolic Orbits

Given t, find r and v

For elliptic orbits:

1. Given time, t, solve for Hyperbolic Mean Anomaly

$$M(t) = \sqrt{\frac{\mu}{-a^3}}t$$

2. Given Mean Anomaly, solve for hyperbolic anomaly

$$M(t) = e\sinh H - H$$

3. Given hyperbolic anomaly, solve for true anomaly

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}, \qquad v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Summary

This Lecture you have learned:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Summary

Properties of Keplerian Orbits

Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis $a < 0$ b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 + b^2 / a^2} \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a\,(1 - e)$	$q = p/2$	$q = a\,(1 - e)$
Velocity, V, at distance, r, from Focus	$V^2 = \mu / r$	$V^2 = \mu\,(2/r - 1/a)$	$V^2 = 2\mu / r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, $\mathcal{E}$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = 0$	$\mathcal{E} = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu / a^3}$	$n = \sqrt{\mu / a^3}$	$n = \sqrt{\mu}$	$n = \sqrt{\mu / (-a)^3}$
Period, P	$P = 2\pi / n$	$P = 2\pi / n$	$P = \infty$	$P = \infty$
Anomaly	$v = M = E$	Eccentric anomaly, E $\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{v}{2} = D/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e \sin E$	$M = qD + (D^3/6)$	$M = (e \sinh F) - F$
Distance from Focus, $r = q\,(1+e)/(1 + e\cos v)$	$r = a$	$r = a\,(1 - e\cos E)$	$r = q + (D^2/2)$	$r = a\,(1 - e\cosh F)$
$r\,dr/dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\,\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\,\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{dv}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$	$\frac{dA}{dt} = \sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$

$\mu = GM$ is the gravitational constant of the central body; v is the true anomaly, and $M \equiv n\,(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion.