Lecture 6: Orientation in Space and The Orbital Plane
In this Lecture, you will learn:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concept: Celestial Coordinate Systems

- The Earth-Centered Inertial reference frame
- The line of nodes

Orientation of the 2D Orbit in 3D space

- How to construct all orbital elements from $\mathbf{r}$ and $\mathbf{v}$
- A Numerical Illustration
So far, all orbits are parameterized by 3 parameters

- semimajor axis, $a$
- eccentricity, $e$
- true anomaly, $f$

- $a$ and $e$ define the geometry of the orbit.
- $f$ describes the position within the orbit (a proxy for time).
The Orbital Elements

But orbits are not 2-dimensional!
The Orbital Elements

**Note:** We have shown how to use $a$, $e$ and $f$ to find the scalars $r$ and $v$.

**Question:** How do we find the vectors $\vec{r}$ and $\vec{v}$?

**Answer:** We have to determine how the orbit is oriented in space.
- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.
Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?
Response: In which coordinate system??

- The origin is the center of the earth
- We need to define the $\hat{x}$, $\hat{y}$, and $\hat{z}$ vectors.
The \( \hat{z} \) vector is defined to be the vector parallel to the axis of rotation of the earth.

- Can apply to other planets
- Does not apply to Heliocentric Coordinates

**Definition 1.**

The **Equatorial Plane** is the set of vectors normal to the axis of rotation.
In fact, the Heliocentric Earth Equatorial (HEEQ/HS) coordinate system uses the mean rotation vector of the sun as the $\hat{z}$ vector and the plane perpendicular as the equatorial plane. The solar central meridian (Sun-Earth line) is the $\hat{x}$ direction. Note that different solar latitudes rotate at different speeds.
The rotation vector of the sun is unreliable.

- In heliocentric coordinates, the \( \hat{z} \) vector is normal to the ecliptic plane.

**Definition 2.**

The **Ecliptic Plane** is the orbital plane of the earth in motion about the sun.

- From the earth, the ecliptic plane is defined by the apparent motion of the sun about the earth.
  - Determined by the location of eclipses (hence the name).
- In heliocentric coordinates, \( \hat{x} \) is either FPOA or the sun-earth vector.
All planets move in the ecliptic plane. If you locate the planets in the night sky, they all form a line.

If they did not, transits and alignments would be exceedingly rare.
The Ecliptic Plane

**Definition 3.**

The **Inclination to the Ecliptic** is the angle between the equatorial and ecliptic planes.

Currently, the inclination to the ecliptic is 23.5°.
To complete the ECI coordinate system, we will define an \( \hat{x} \) axis in the equatorial plane.

The \( \hat{y} \) axis is then given by the right-hand rule.

A fixed location for \( \hat{x} \) is the intersection of the equatorial and ecliptic planes. But there are two such points, at the two equinoxes (vernal and autumnal).

The **First Point of Aries** is the earth-sun vector at the vernal equinox.

**Question** Does the FPOA lie at the ascending or descending node?
Notice the tilt of the earth is perpendicular to the earth-sun vector at equinox.
ECI: The First Point in Aries

- The First Point of Aries is so named because this direction used to point towards the Constellation Aries.
- Precession of the earth’s rotation vector means the FPOA now actually points toward Pisces.

- Since Motion of the FPOA is caused be precession, its motion is **Periodic**, not Secular.
  - The Period is about 26,000 years.
- The Coordinate System is not truly inertial.
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- 1 degree every 72 years
Summary: The ECI frame

- \( \hat{z} \) - North Pole
- \( \hat{x} \) - FPOA
- \( \hat{y} \) - Right Hand Rule

Because the FPOA migrates with time, positions given in ECI must be referenced to a year.

- **J2000** - frame as defined at 12:00 TT on Jan 1, 2000.
- **TOD** - True of Data: date is listed explicitly.
Note there are many other reference frames of interest

- Earth Centered Earth Fixed (ECEF)
- Perifocal
- Frenet System (Satellite Normal, Drag)
- Gaussian (Satellite Radial)
- Topocentric Horizon
- Topocentric Equatorial

We will return to some of these frames when necessary.
Other Reference Frames

- ECEF - Fundamental plane is equatorial. Reference direction is Prime meridian.
- Perifocal - fundamental plane is orbital plane. reference direction is eccentricity vector.
- Frenet - fundamental plane is orbital plane. Reference direction is velocity vector.
- Satellite radial - fundamental plane is orbital plane. Reference direction is earth-satellite vector.
- The Horizontal coordinate system is similar to the topocentric horizon. It uses altitude and azimuth (usually measured from north to east).
- Heliocentric Earth Ecliptic (HEE) - Earth orbital plane and Sun-earth Vector
- Heliocentric Ares Ecliptic (HAE) - Earth orbital plane and vernal equinox (Equatorial/Ecliptic intersection)
- Heliocentric Earth Equatorial (HEEQ/HS) - Solar Equator and solar central meridian.
- Galactic - Galactic Plane and Sun-Galactic center vector (sun-centered)
Orbital Elements

Now that we have our coordinate system,

**Question:** Suppose we are given $\vec{r}$ and $\vec{v}$ in the ECI frame. How to describe the orientation of the orbit?

**Answer:** 3 new orbital elements.
- Inclination
- Right Ascension
- Argument of Periapse
Angle the orbital plane makes with the reference plane at ascending node. The orbit is

- **Prograde** if $0 < i < 90^\circ$.
- **Retrograde** if $90 < i < 180^\circ$. 
Retrograde orbits move counter to earth’s rotation, so seem faster when viewed from the earth.
Inclination can be found from $\vec{h}$ as

$$\vec{h} \cdot \hat{z} = h \cos i.$$ 

- If $\vec{h}$ is defined in ECI, then $i = \cos^{-1} \left( \frac{h_3}{h} \right)$.
- No quadrant ambiguity because by definition, $i \leq 180$ deg
An important vector in defining the orbit is the line of nodes.

**Definition 4.**

The **Line of Nodes** is the vector pointing to where the satellite crosses the equatorial plane from the southern to northern hemisphere.

\[ \vec{n} = \hat{z} \times \vec{h} \]
The Line Of Nodes

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Definition 4.

The **Line of Nodes** is the vector pointing to where the satellite crosses the equatorial plane from the southern to northern hemisphere.

\[ \vec{n} = \hat{z} \times \vec{h} \] forces \( \vec{n} \) to lie in both the orbital plane and equatorial plane.

**Question:** What would be the formula if we wanted \( \vec{n} \) to point to the **Descending Node**
The Line Of Nodes

- Lies at the intersection of the equatorial and orbital planes.
- Points toward the Ascending Node.
- Zero for equatorial orbits \((i = 0)\).
The Orbital Plane
Right Ascension of Ascending Node, $\Omega$

The Angle measured from reference direction, $\hat{x}$ in the reference plane to ascending node.

- Defined to be $0 \leq \Omega \leq 360$
- Undefined for equatorial orbits ($i = 0$).
The Orbital Plane

Right Ascension of Ascending Node, $\Omega$

RAAN can be found from the line of nodes as

$$\cos(\Omega) = \frac{\hat{x} \cdot \vec{n}}{||\vec{n}||}$$

Must resolve quadrant ambiguity.

**Quadrant Ambiguity:** Calculators assume $\Omega$ is in quadrant 1 or 2. Correct as

$$\Omega = \begin{cases} 
\Omega & \hat{y} \cdot \vec{n} \geq 0 \\
360 - \Omega & \hat{y} \cdot \vec{n} < 0
\end{cases}$$
Argument of Periapse, $\omega$

- Undefined for *Circular* Orbits.
- Define so $0 \leq \omega < 360 \text{ deg}$

**Definition 5.**

The **Argument of Periapse** is the angle from line of nodes to the point of periapse.
Argument of Periapse, $\omega$

Can be calculated from

$$\cos(\omega) = \frac{\vec{n} \cdot \vec{e}}{||\vec{n}||e}$$

Must resolve quadrant ambiguity

**Quadrant Ambiguity:** Calculators assume $\omega$ is in quadrant 1 or 2. Correct as

$$\omega = \begin{cases} 
  \omega & \hat{z} \cdot \vec{e} \geq 0 \\
  360 - \omega & \hat{z} \cdot \vec{e} < 0 
\end{cases}$$
Argument of Periapse, $\omega$

Quadrant check determines if $\vec{e}$ is in southern or northern hemisphere.
True Anomaly, $f$ (sometimes $\nu$)

Can be calculated directly from the polar equation

$$r = \frac{p}{1 + e \cos f}$$

$$f = \cos^{-1} \left( \frac{p - r}{re} \right)$$

Or can be calculated from

$$\cos(f) = \frac{\vec{r} \cdot \vec{e}}{\|\vec{r}\| \|e\|}$$

In BOTH CASES, we have quadrant ambiguity

**Quadrant Ambiguity:** Is $\|\vec{r}\|$ getting longer or shorter?

$$f = \begin{cases} f & \hat{r} \cdot \vec{v} \geq 0 \\ 360 - f & \hat{r} \cdot \vec{v} < 0 \end{cases}$$
True Anomaly, $f$ (sometimes $\nu$)

Can be calculated directly from the polar equation:

$$r = \frac{p}{1 + e \cos f}$$

Or can be calculated from:

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In BOTH CASES, we have quadrant ambiguity:

Quadrant Ambiguity: Is $\|\vec{r}\|$ getting longer or shorter?

$$f = \begin{cases} f & \vec{r} \cdot \vec{v} \geq 0 \\ 360 - f & \vec{r} \cdot \vec{v} < 0 \end{cases}$$

$r < 0$ when $f > 180^\circ$
Summary: Visualization
Equinoctial elements avoid singularities caused by circular and equitorial orbits. In this system, $i$ and $\omega$ are replaced by new elements. Used by Lagrange.
Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position

\[ \vec{r} = \begin{bmatrix} 6524.8 \\ 6862.8 \\ 6448.3 \end{bmatrix} \text{ km} \]

moving with velocity

\[ \vec{v} = \begin{bmatrix} 4.901 \\ 5.534 \\ -1.976 \end{bmatrix} \text{ km/s} \]

Determine the orbital elements.

Solution: Although not necessary, as per your homework, let’s first convert to canonical units (1\(ER = 6378.14\) km, 1\(TU = 806.3\) s).

\[ \vec{r}' = \frac{\vec{r}}{6378.14\text{ km}} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} \]

\[ \vec{v}' = \frac{\vec{v}}{806.8\text{ s}} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix} \]

First, let’s construct angular momentum, \(\vec{h}\), the line of nodes, \(\vec{n}\) and the eccentricity vector, \(\vec{e}\).
Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position
\[ \vec{r} = \begin{bmatrix} 6224.8 \\ 6962.8 \\ 6445.3 \end{bmatrix} \text{ km} \] moving with velocity
\[ \vec{v} = \begin{bmatrix} 4.901 \\ 5.534 \\ -1.976 \end{bmatrix} \text{ km/s} \]

Determine the orbital elements.

Solution: Although not necessary, as per your homework, let's first convert to canonical units (\(1 \text{ ER} = 6378.14 \text{ km}, 1 \text{ TU} = 806.8 \text{ s}\)).

\[ \vec{r}' = \frac{\vec{r}}{6378.14} = \begin{bmatrix} 1.021 \\ 1.076 \\ 1.011 \end{bmatrix} \]
\[ \vec{v}' = \frac{\vec{v}}{806.8} = \begin{bmatrix} 0.062 \\ 0.067 \\ -0.025 \end{bmatrix} \]

First, let's construct angular momentum, \(\vec{h}\), the line of nodes, \(\vec{n}\) and the eccentricity vector, \(\vec{e}\).

Question: What if you were given \(\vec{r}\) and \(\vec{v}\) in the ECEF coordinate system?
We construct $\vec{h}$, $\vec{n}$ and $\vec{e}$.

$$\vec{h} = \vec{r} \times \vec{v} = \begin{bmatrix} -0.9767 \\ 0.882 \\ 0.049 \end{bmatrix} \frac{ER^2}{TU}$$

Since $\vec{r}$ and $\vec{v}$ are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h} = \begin{bmatrix} -0.882 \\ -0.9767 \\ 0 \end{bmatrix} \frac{ER^2}{TU}.$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r} = \begin{bmatrix} -0.315 \\ -0.385 \\ 0.668 \end{bmatrix}$$

where recall $\mu = 1$ in canonical units.
Example: Finding Orbital Elements

Continued

Now we begin solving for orbital elements.

\[ e = \| \vec{e} \| = 0.8328 \]

Use energy to calculate \( a \).

\[ E = \frac{v^2}{2} - \frac{\mu}{r} = -0.088 \]

\[ a = -\frac{\mu}{2E} = 5.664ER \]

\[ p = \frac{h^2}{\mu} = 1.735ER \]

We can now calculate our three new orbital elements as indicated. Start with inclination

\[ i = \cos^{-1} \left( \frac{\vec{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{h} \right) = 87.9 \text{ deg} \]

No quadrant ambiguity by definition.
Continue with RAAN, we want the angle between $\hat{x}$ and $\vec{n}$.

$$\Omega = \cos^{-1} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right) = \pm 132.10 \text{ deg}$$

Because $\cos$ has quadrant ambiguity, we must check the quadrant. Specifically, we need the sign of

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \vec{n} = -.9767 < 0$$

Therefore, $\vec{n}$ is in the third quadrant, and we need to correct

$$\Omega = 360 - 132.10 = 227.9 \text{ deg}$$
Next, the argument of perigee is the angle between $\vec{e}$ and $\vec{n}$.

$$\omega = \cos^{-1}\left(\frac{\vec{n} \cdot \vec{e}}{||\vec{n}||e}\right) = \pm 53.4 \text{ deg}$$

We resolve the quadrant ambiguity by checking

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \vec{e} = .668 > 0$$

so we are in the right quadrant

$$\omega = 53.4 \text{ deg}$$
Finally, we solve for true anomaly. But this is simply the angle between $\vec{r}$ and $\vec{e}$, so we can use

$$f = \cos^{-1}\left( \frac{\vec{r} \cdot \vec{e}}{re} \right) = \pm 92.3 \text{ deg}$$

We resolve the quadrant ambiguity by checking

$$\vec{r} \cdot \vec{v} > 0$$

So we are in the right quadrant

$$f = 92.3 \text{ deg}$$
Finally, we solve for true anomaly. But this is simply the angle between $\vec{r}$ and $\vec{r}$, so we can use 

$$f = \cos^{-1} \left( \frac{\vec{r} \cdot \vec{r}}{|\vec{r}|^2} \right) = \pm 92.3 \text{ deg}.$$ 

We resolve the quadrant ambiguity by checking 

$$\vec{r} \cdot \vec{v} > 0$$ 

So we are in the right quadrant. 

$$f = 92.3 \text{ deg}.$$ 

- The quadrant ambiguity here is a bit tricky to visualize. 
- From perigee to apogee, the spacecraft is getting farther from the planet, meaning the velocity vector has a positive outward component. From apogee to perigee, the spacecraft is getting uniformly closer to the planet, meaning that velocity vector is pointing slightly inwards.
Summary

This Lecture you have learned:

The Orbital Plane
- Inclination
- Right Ascension
- Argument of Periapse

New Concepts
- The Earth-Centered Inertial reference frame
- The line of nodes

Practice
- How to construct all orbital elements from $\vec{r}$ and $\vec{v}$
- A Numerical Illustration