Spacecraft Dynamics and Control

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Lecture 6: Orientation in Space and The Orbital Plane

Introduction

In this Lecture, you will learn:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concept: Celestial Coordinate Systems

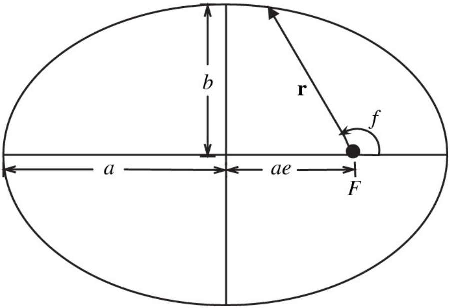
- The Earth-Centered Inertial reference frame
- The line of nodes

Orientation of the 2D Orbit in 3D space

- How to construct all orbital elements from $\vec{r}$ and $\vec{v}$
- A Numerical Illustration

So far, all orbits are parameterized by 3 parameters

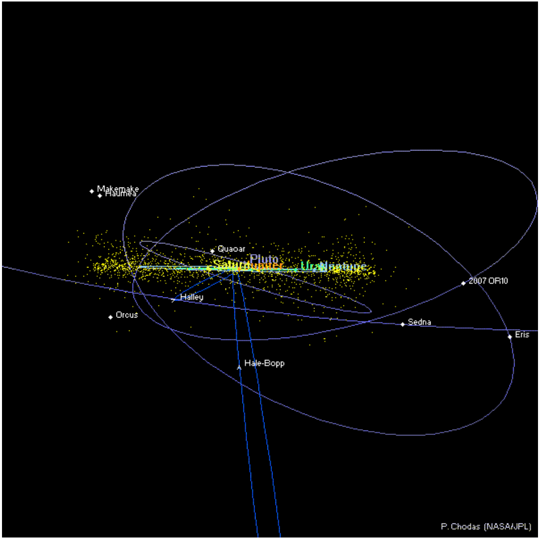
- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

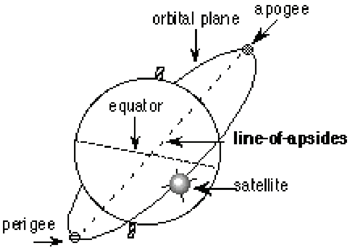
The Orbital Elements

But orbits are not 2-dimensional!



The Orbital Elements

Note: We have shown how to use a, e and f to find the scalars r and v.



Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.

The Orbital Elements

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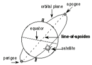

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Answer: We have to determine how the orbit is oriented in space.

- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
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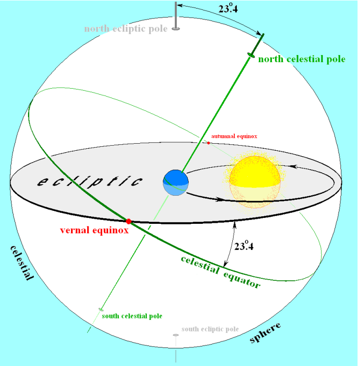
  - Orientation can be determined by 3 rotations.

- Although $\vec{e}$ and $\vec{h}$ represent 6 components, we only actually need three. The $\vec{h}$ represents orientation of the orbital plane, and so we don't care about the roll axis in the classic 1-2-3 rotation matrices. That is, this orientation has symmetry about the angular momentum vector. The eccentricity vector is always perpendicular to the angular momentum vector, which gives one constraint. The second is that its length equals the eccentricity of the orbit. This leaves a single degree of freedom.

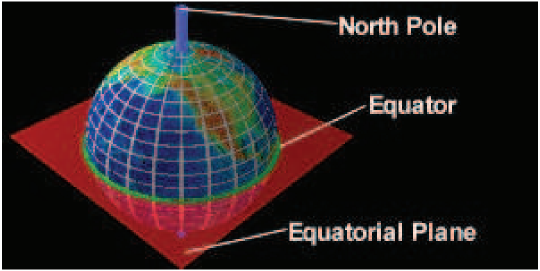
Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?
Response: In which coordinate system??



- The origin is the center of the earth
- We need to define the $\hat{x}$, $\hat{y}$, and $\hat{z}$ vectors.

ECI: The Equatorial Plane

Defining the $\hat{z}$ vector



- The $\hat{z}$ vector is defined to be the vector parallel to the axis of rotation of the earth.
- Can apply to other planets
- Does not apply to Heliocentric Coordinates (axis hard to measure)

Definition 1.

The **Equatorial Plane** is the set of vectors normal to the axis of rotation.

ECI: The Equatorial Plane

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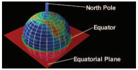

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Definition 1.
The **Equatorial Plane** is the set of vectors normal to the axis of rotation.

- In fact, the Heliocentric Earth Equatorial (HEEQ/HS) coordinate system uses the mean rotation vector of the sun as the $\hat{z}$ vector and the plane perpendicular as the equatorial plane. The solar central meridian (Sun-Earth line) is the $\hat{x}$ direction. Note that different solar latitudes rotate at different speeds.

The Ecliptic Plane

Heliocentric Coordinates

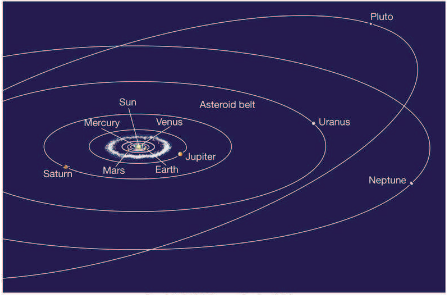
The rotation vector of the sun is unreliable.

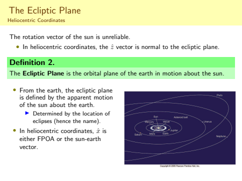
- In heliocentric coordinates, the $\hat{z}$ vector is normal to the ecliptic plane.

Definition 2.

The **Ecliptic Plane** is the orbital plane of the earth in motion about the sun.

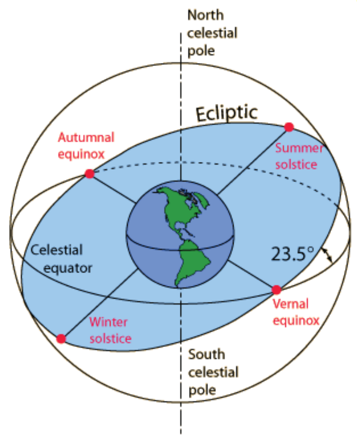
- From the earth, the ecliptic plane is defined by the apparent motion of the sun about the earth.
  - Determined by the location of eclipses (hence the name).
- In heliocentric coordinates, $\hat{x}$ is either FPOA or the sun-earth vector.





- All planets move approximately in the ecliptic plane. If you locate the planets in the night sky, they all form a line ($\pm 6^\circ$) - the zodiac.
- However, an alternative fundamental plane is "Laplace's invariable plane", which is defined by the angular momentum vector of the 9-body solar system. The difference between the invariable plane and ecliptic plane is around 5°
- All transits and eclipses occur in the ecliptic plane.

The Ecliptic Plane



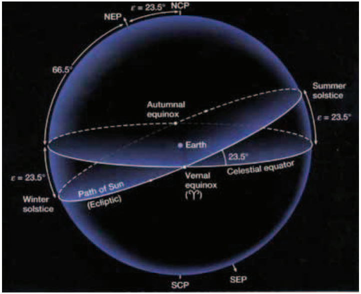
Definition 3.

The **Inclination to the Ecliptic** is the angle between the equatorial and ecliptic planes.

Currently, the inclination to the ecliptic is $23.5\,\mathrm{deg}$.

ECI: The First Point of Aries

- To complete the ECI coordinate system, we will define an $\hat{x}$ axis in the equatorial plane.
- The $\hat{y}$ axis is then given by the right-hand rule.



A fixed location for $\hat{x}$ is the intersection of the equatorial and ecliptic planes. But there are two such points, at the two equinoxes (vernal and autumnal)

The **First Point of Aries** is the earth-sun vector at the vernal equinox.

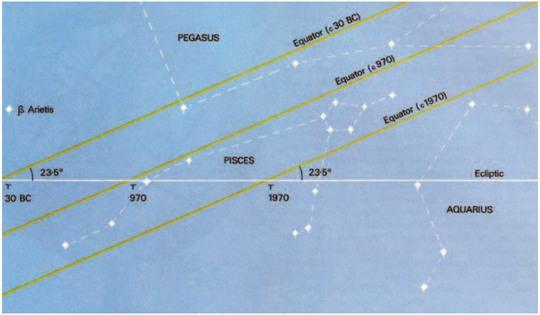
Question Does the FPOA lie at the ascending or descending node?

Celestial Sphere

Notice the tilt of the earth is perpendicular to the earth-sun vector at equinox.

ECI: The First Point in Aries

- The First Point of Aries is so named because this direction used to point towards the Constellation Aries.
- Precession of the earth's rotation vector means the FPOA now actually points toward Pisces.



- Since Motion of the FPOA is caused by precession, its motion is **Periodic**, not Secular.
  - The Period is about 26,000 years.
- The Coordinate System is not truly inertial.

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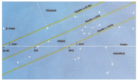
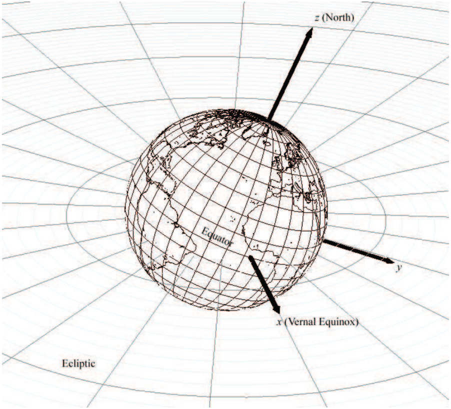


- Since Motion of the FPOA is caused by precession, its motion is **Periodic**, not Secular.
  - The Period is about 26,000 years.
- The Coordinate System is not truly inertial.

- 1 degree every 72 years

Summary: The ECI frame

- $\hat{z}$ - North Pole
- $\hat{x}$ - FPOA
- $\hat{y}$ - Right Hand Rule



Because the FPOA migrates with time, positions given in ECI must be referenced to a year

- **J2000** - frame as defined at 12:00 TT on Jan 1, 2000.
- **TOD** - True of Date: date is listed explicitly.

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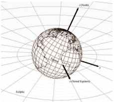

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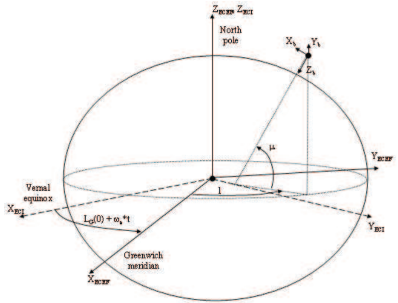
The motion of the FPOA is due to both movement of the ecliptic plane and the celestial equator.

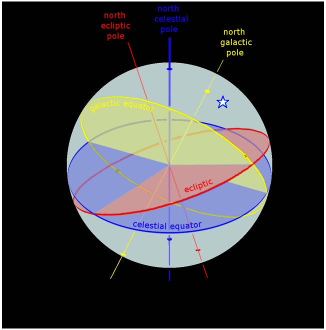
- The celestial equator moves much more because it is directly defined by the rotation of the Earth, which precesses.
- The motion of the ecliptic

Variations on ECI:

- **TOD:** ECI Coordinate system as defined by FPOA on given date
- **J2000:** ECI Coordinate system as defined by FPOA at 12:00 Terrestrial time (similar to GMT or UTC) on Jan 1, 2000
- **MOD:** (Mean of Date) Same as TOD, but averages out the nutation (not precession) as computed on a specific date.
- **M50:** (Mean of Date) Same as J2000, but uses 1950 and averages out the nutation (not precession).
- **GCRF:** (Geocentric Celestial) Similar to ECI, but $\hat{z}$ is the north ecliptic pole.

Other Reference Frames



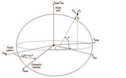


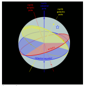
Note there are many other reference frames of interest

- Earth Centered Earth Fixed (ECEF)
- Perifocal
- Frenet System (Satellite Normal, Drag)
- Gaussian (Satellite Radial)
- Topocentric Horizon
- Topocentric Equatorial

We will return to some of these frames when necessary.

Other Reference Frames





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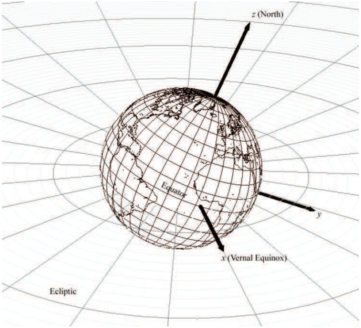
- ECEF - Fundamental plane is equatorial. Reference direction is Prime meridian.
- Perifocal - fundamental plane is orbital plane. reference direction is eccentricity vector.
- Frenet - fundamental plane is orbital plane. Reference direction is velocity vector.
- Satellite radial - fundamental plane is orbital plane. Reference direction is earth-satellite vector.
- The Horizontal coordinate system is similar to the topocentric horizon. It uses altitude and azimuth (usually measured from north to east).
- Heliocentric Earth Ecliptic (HEE) - Earth orbital plane and Sun-earth Vector
- Heliocentric Ares Ecliptic (HAE) - Earth orbital plane and vernal equinox (Equatorial/Ecliptic intersection)
- Heliocentric Earth Equatorial (HEEQ/HS) - Solar Equator and solar central meridian.
- Barycentric Celestial Reference Frame (or now ICRS) - Origin is Barycenter of solar system. $\hat{z}$ is the celestial north pole. $\hat{x}$ is approximately FPOA in J2000, but fixed relative to quasars.
- Galactic - Galactic Plane and Sun-Galactic center vector (sun-centered)

Orbital Elements

Now that we have our coordinate system,

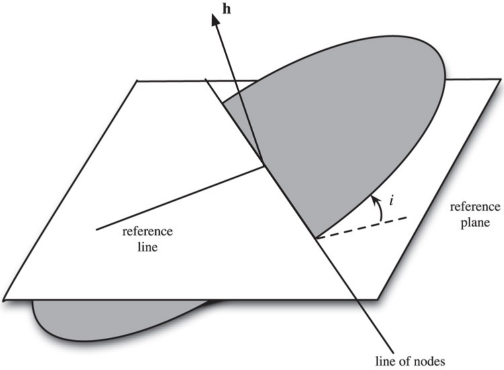


Question:, Suppose we are given $\vec{r}$ and $\vec{v}$ in the ECI frame. How to describe the orientation of the orbit?

Answer: 3 new orbital elements.

- Inclination
- Right Ascension
- Argument of Periapse

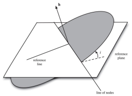
Inclination, i



Angle the orbital plane makes with the reference plane.

- As measured at the **Ascending Node**
- Note the orbit is counterclockwise about the angular momentum vector.
- Obeys the RHR.

Inclination, i



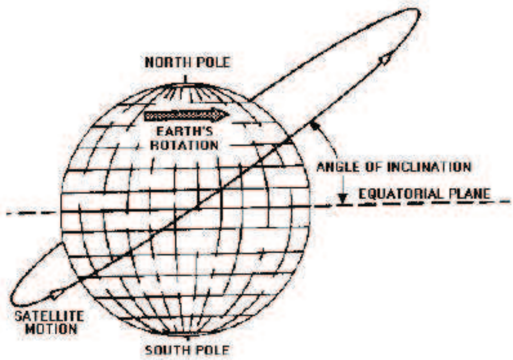
Angle the orbital plane makes with the reference plane.

- As measured at the **Ascending Node**
- Note the orbit is counterclockwise about the angular momentum vector.
- Obeys the RHR.

- Think of the 2D orbit in space. Z-axis is out of the ecliptic plane. X-axis is line of nodes.
- first rotation is about the line of nodes.

The Orbital Plane

Inclination, i

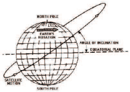


Angle the orbital plane makes with the reference plane at ascending node. The orbit is

- **Prograde** if $0° < i < 90°$.
- **Retrograde** if $90° < i < 180°$.

The Orbital Plane

Inclination, i



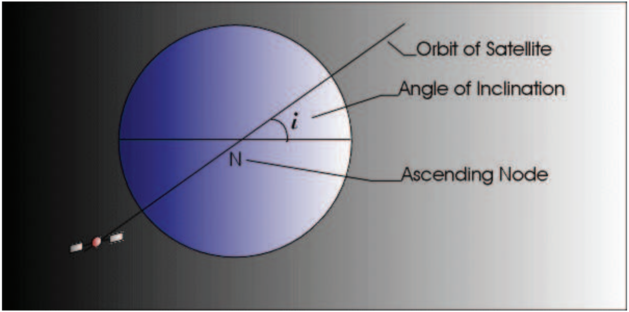
Angle the orbital plane makes with the reference plane at ascending node. The orbit is

- **Prograde** if $0^\circ < i < 90^\circ$.
- **Retrograde** if $90^\circ < i < 180^\circ$.

Retrograde orbits move counter to earth's rotation, so seem faster when viewed from the earth.

- Its a polar orbit if $i = 90^\circ$
- Its a equatorial orbit if $i = 0^\circ$

The Orbital Plane
Inclination, i

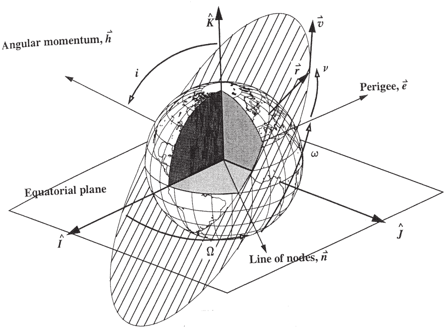


Inclination can be found from $\vec{h}$ as

$$\vec{h} \cdot \hat{z} = h \cos i.$$

- If $\vec{h}$ is defined in ECI, then $i = \cos^{-1} \frac{h_3}{h}$.
- No quadrant ambiguity because by definition, $i \leq 180°$
  - If $i > 180°$, the ascending node becomes the descending node.

The Line Of Nodes



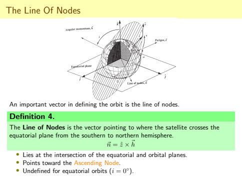
An important vector in defining the orbit is the line of nodes.

Definition 4.

The **Line of Nodes** is the vector pointing to where the satellite crosses the equatorial plane from the southern to northern hemisphere.

$$\vec{n} = \hat{z} \times \vec{h}$$

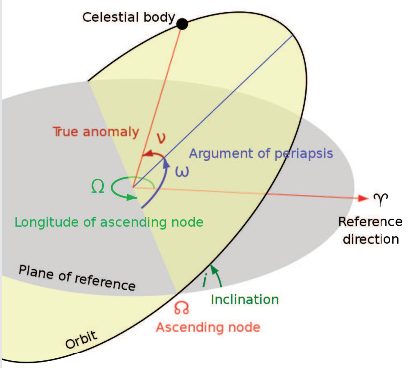
- Lies at the intersection of the equatorial and orbital planes.
- Points toward the Ascending Node.
- Undefined for equatorial orbits ($i = 0°$).



$\vec{n} = \hat{z} \times \vec{h}$ forces $\vec{n}$ to lie in both the orbital plane and equatorial plane.

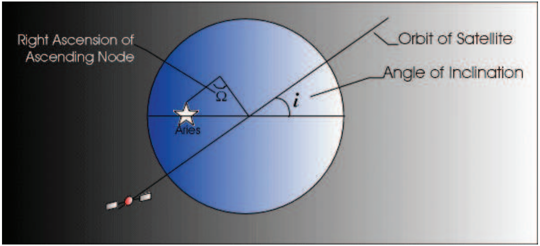
Question: What would be the formula if we wanted $\vec{n}$ to point to the *Descending Node*

Answer: Either $\vec{h} \times \hat{z}$ or $-\hat{z} \times \vec{h}$



The Orbital Plane

Right Ascension of Ascending Node, Ω

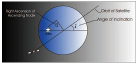


The Angle measured from reference direction, $\hat{x}$ in the reference plane to *ascending node*.

- Defined to be $0^\circ \leq \Omega \leq 360^\circ$
- Undefined for equatorial orbits ($i = 0^\circ$).

The Orbital Plane

Right Ascension of Ascending Node, Ω



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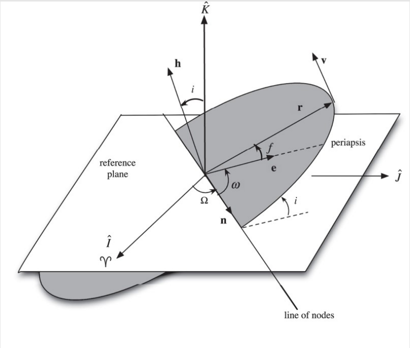


Angle measured from reference direction in the reference plane to intersection with orbital plane.

- Second rotation is about the Z-axis.

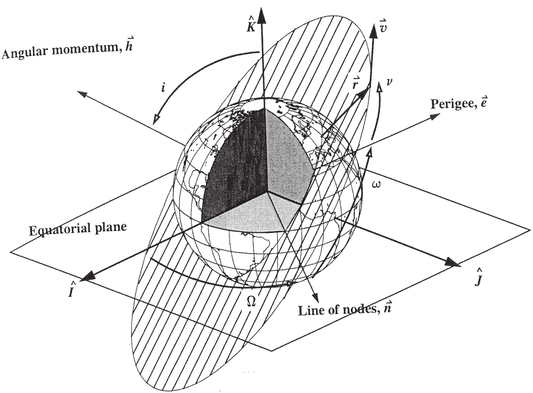
The Orbital Plane

Right Ascension of Ascending Node, Ω

RAAN can be found from the line of nodes as

$$\cos(\Omega) = \frac{\hat{x} \cdot \vec{n}}{\|\vec{n}\|}$$

Must resolve quadrant ambiguity.

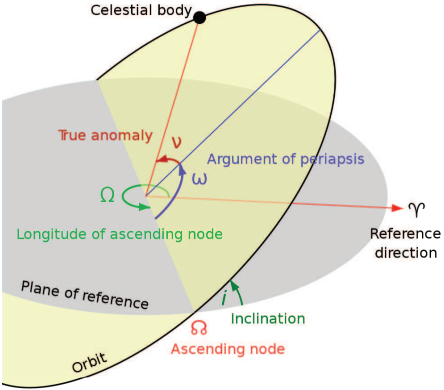


Quadrant Ambiguity: Calculators assume Ω is in quadrant 1 or 2. Correct as

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360° - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

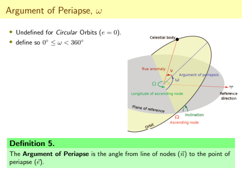
Argument of Periapse, ω

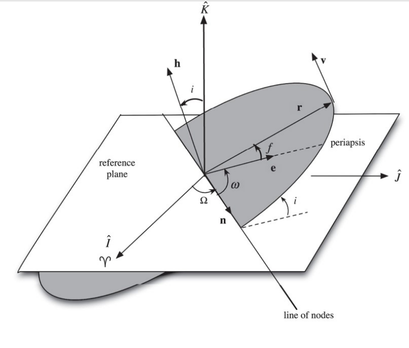
- Undefined for *Circular* Orbits ($e = 0$).
- define so $0^\circ \leq \omega < 360^\circ$



Definition 5.

The **Argument of Periapse** is the angle from line of nodes ($\vec{n}$) to the point of periapse ($\vec{e}$).





Angle measured from reference plane to point of periapse ($\vec{e}$).

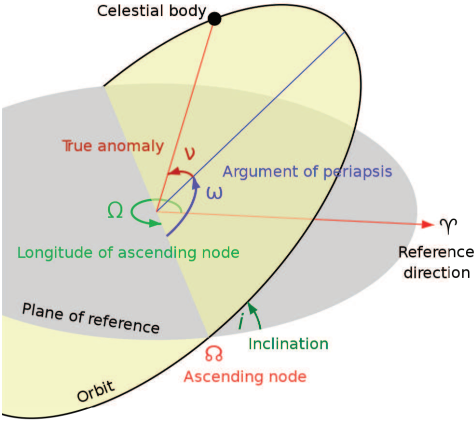
- Third rotation is about angular momentum vector.

Argument of Periapse, ω

Can be calculated from

$$\cos(\omega) = \frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\| e}$$

Must resolve quadrant ambiguity



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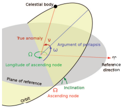

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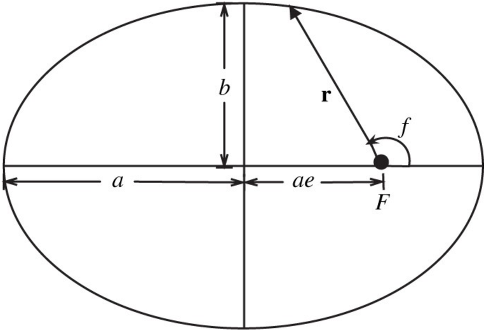
Quadrant check determines if $\vec{e}$ is in southern or northern hemisphere.

True Anomaly, f (sometimes ν)

Can be calculated directly from the polar equation

$$r = \frac{p}{1 + e\cos f}$$

$$f = \cos^{-1}\left(\frac{p-r}{re}\right)$$



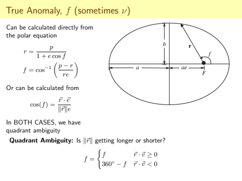
Or can be calculated from

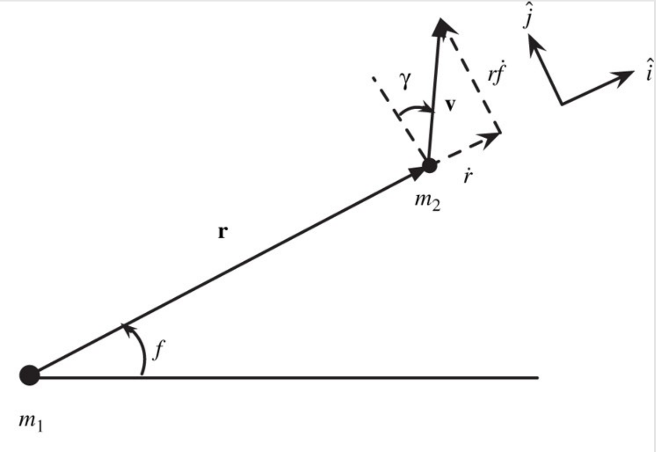
$$\cos(f) = \frac{\vec{r}\cdot\vec{e}}{\|\vec{r}\|e}$$

In BOTH CASES, we have quadrant ambiguity

Quadrant Ambiguity: Is $\|\vec{r}\|$ getting longer or shorter?

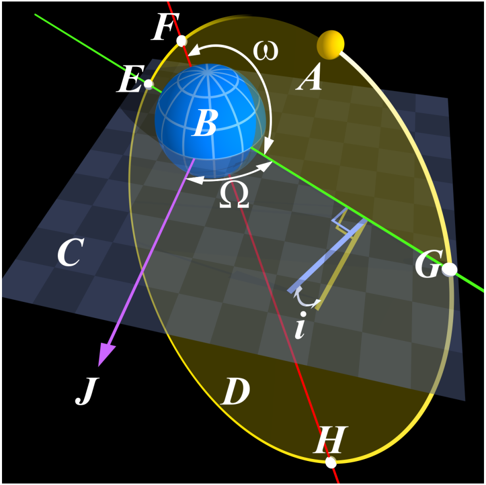
$$f = \begin{cases} f & \vec{r}\cdot\vec{v} \geq 0 \\ 360^\circ - f & \vec{r}\cdot\vec{v} < 0 \end{cases}$$



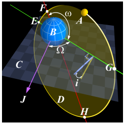


$\dot{r} < 0$ when $f > 180^\circ$

Summary: Visualization



Summary: Visualization



- Equinoctial elements avoid singularities caused by circular and equitorial orbits and are also valid for hyperbolic orbits. In this system, i and ω are replaced by new elements. Useful in perturbation analysis.

Modified Equinoctial elements:

- $p = a(1 - e^2)$
- $d = e\sin(\omega + \Omega)$
- $g = e\cos(\omega + \Omega)$
- $h = \tan(i/2)\cos\Omega$
- $k = \tan(i/2)\sin\Omega$
- $L = \Omega + \omega + f$

For circular orbits, $d = g = 0$
For equatorial orbits, $h = k = 0$

Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position

$$\vec{r} = \begin{bmatrix}6524.8\\6862.8\\6448.3\end{bmatrix} km \qquad \text{moving with velocity} \qquad \vec{v} = \begin{bmatrix}4.901\\5.534\\-1.976\end{bmatrix} km/s$$

Determine the orbital elements.

Solution: Although not necessary, as per your homework, lets first convert to canonical units ($1ER = 6378.14km$, $1TU = 806.3s$).

$$\vec{r}' = \frac{\vec{r}}{6378.14km} = \begin{bmatrix}1.023\\1.076\\1.011\end{bmatrix}$$

$$\vec{v}' = \vec{v}\frac{806.8s}{6378.14km} = \begin{bmatrix}.62\\.7\\-.25\end{bmatrix}$$

First, lets construct angular momentum, $\vec{h}$, the line of nodes, $\vec{n}$ and the eccentricity vector, $\vec{e}$.

Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position

$$\vec{r} = \begin{bmatrix} 6524.8 \\ 6862.8 \\ 6448.3 \end{bmatrix} km \qquad \text{moving with velocity} \qquad \vec{v} = \begin{bmatrix} 4.901 \\ 5.534 \\ -1.976 \end{bmatrix} km/s$$

Determine the orbital elements.

Solution: Although not necessary, as per your homework, lets first convert to canonical units ($1ER = 6378.14km$, $1TU = 806.3s$).

$$\vec{r}' = \frac{\vec{r}}{6378.14km} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix}$$

$$\vec{v}' = \vec{v}\frac{806.8s}{6378.14km} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix}$$

First, lets construct angular momentum, $\vec{h}$, the line of nodes, $\vec{n}$ and the eccentricity vector, $\vec{e}$.

Question: What if you were given $\vec{r}$ and $\vec{v}$ in the ECEF coordinate system?

Example: Finding Orbital Elements

Continued

We construct $\vec{h}$, $\vec{n}$ and $\vec{e}$.

$$\vec{h} = \vec{r} \times \vec{v} = \begin{bmatrix} -.9767 \\ .882 \\ .049 \end{bmatrix} \frac{ER^2}{TU}$$

Since $\vec{r}$ and $\vec{v}$ are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h} = \begin{bmatrix} -.882 \\ -.9767 \\ 0 \end{bmatrix} \frac{ER^2}{TU}.$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r} = \begin{bmatrix} -.315 \\ -.385 \\ .668 \end{bmatrix}$$

where recall $\mu = 1$ in canonical units.

Example: Finding Orbital Elements

Continued

Now we begin solving for orbital elements.

$$e = \|\vec{e}\| = .8328$$

Use energy to calculate a.

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -.088$$
$$a = -\frac{\mu}{2E} = 5.664ER$$
$$p = \frac{h^2}{\mu} = 1.735ER$$

We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1}\left(\frac{\vec{h}}{h} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 87.9^\circ$$

No quadrant ambiguity by definition.

Example: Finding Orbital Elements

Continued

Continue with RAAN, we want the angle between $\hat{x}$ and $\vec{n}$.

$$\Omega = \cos^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|}\right) = \pm 132.10^\circ$$

Because $\cos$ has quadrant ambiguity, we must check the quadrant. Specifically, we need the sign of

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \vec{n} = -.9767 < 0$$

Therefore, $\vec{n}$ is in the *third* quadrant, and we need to correct

$$\Omega = 360^\circ - 132.10^\circ = 227.9^\circ$$

Example: Finding Orbital Elements

Continued

Next, the argument of perigee is the angle between $\vec{e}$ and $\vec{n}$.

$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{\|\vec{n}\|e}\right) = \pm 53.4^\circ$$

We resolve the quadrant ambiguity be checking

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\vec{e} = .668 > 0$$

so we are in the right quadrant

$$\omega = 53.4^\circ$$

Finally, we solve for true anomaly. But this is simply the angle between $\vec{r}$ and $\vec{e}$, so we can use

$$f = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right) = \pm 92.3^\circ$$

We resolve the quadrant ambiguity by checking

$$\vec{r}\cdot\vec{v} > 0$$

So we are in the right quadrant

$$f = 92.3^\circ$$

Example: Finding Orbital Elements

Continued

Finally, we solve for true anomaly. But this is simply the angle between $\vec{r}$ and $\vec{e}$, so we can use

$$f = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right) = \pm 92.3^\circ$$

We resolve the quadrant ambiguity by checking

$$\vec{r}\cdot\vec{v} > 0$$

So we are in the right quadrant

$$f = 92.3^\circ$$

- The quadrant ambiguity here is a bit tricky to visualize.
- From perigee to apogee, the spacecraft is getting farther from the planet, meaning the velocity vector has a positive outward component. From apogee to perigee, the spacecraft is getting uniformly closer to the planet, meaning that velocity vector is pointing slightly inwards.

Summary

This Lecture you have learned:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concepts

- The Earth-Centered Inertial reference frame
- The line of nodes

Practice

- How to construct all orbital elements from $\vec{r}$ and $\vec{v}$
- A Numerical Illustration