Spacecraft Dynamics and Control

Matthew M. Peet

Lecture 7: Converting to/from $\vec{r}$ and $\vec{v}$

Introduction

In this Lecture, you will learn:

How to convert between

- a, e, i, Ω, ω, f
- $\vec{r}$ and $\vec{v}$

How to translate $\vec{r}$ and $\vec{v}$ into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Moving the Orbit Forward in Time (Propagation)

Question: If I observe a satellite at 12:00 at position $\vec{r}$ with velocity $\vec{v}$, where will it be at time 15:00?

3 Steps

1. Calculate the orbital elements at 12:00
2. Determine f at 15:00.
3. Convert orbital elements to $\vec{r}$ and $\vec{v}$ at 15:00.

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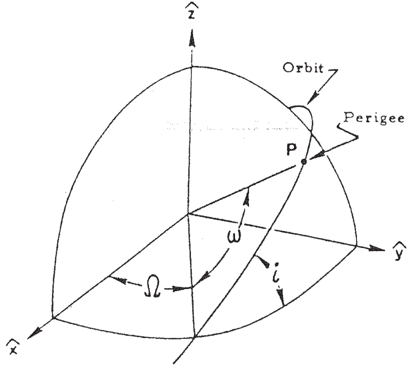
2. Determine f at 15:00.
3. Convert orbital elements to $\vec{r}$ and $\vec{v}$ at 15:00.

- We already know how to do steps 1 and 2.
- Step 3 is more challenging, although the answer is actually simpler.

Finding the Orbital Elements



In the previous lecture, we introduced three new orbital elements.

- Inclination, i
- RAAN, Ω
- Argument of Periapse, ω

We gave a numerical example to illustrate how to find these new elements

Finding the Orbital Elements

Summary

Step 1: Construct $\vec{h}$, $\vec{n}$ and $\vec{e}$.

$$\vec{h} = \vec{r} \times \vec{v}$$

Assume $\vec{r}$ and $\vec{v}$ are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h}$$

$$\vec{e} = \frac{1}{\mu}\vec{v} \times \vec{h} - \frac{\vec{r}}{r}$$

Alternatively

$$\vec{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r} \cdot \vec{v})\,\vec{v}\right]$$

Calculate the scalars $r = \|\vec{r}\|$, $v = \|\vec{v}\|$, $e = \|\vec{e}\|$, $h = \|\vec{h}\|$.

Finding the Orbital Elements

Step 2: Calculate the 2D elements.

$$E = \frac{v^2}{2} - \frac{\mu}{r} \qquad \text{and} \qquad a = -\frac{\mu}{2E}$$

Step 3: Calculate the 3D elements. We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1}\left(\frac{\vec{h}}{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

No quadrant ambiguity.

$$\Omega = \cos^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|}\right)$$

Correct for quadrant.

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360 - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

Finding the Orbital Elements

Argument of perigee is the angle between $\vec{e}$ and $\vec{n}$.

$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{\|\vec{n}\|e}\right)$$

We resolve the quadrant ambiguity be checking

$$\omega = \begin{cases}\omega & \hat{z}\cdot\vec{e} \geq 0 \\ 360-\omega & \hat{z}\cdot\vec{e} < 0\end{cases}$$

True anomaly is the angle between $\vec{r}$ and $\vec{e}$.

$$f = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right)$$

We resolve the quadrant ambiguity by checking

$$f = \begin{cases}f & \vec{r}\cdot\vec{v} \geq 0 \\ 360-f & \vec{r}\cdot\vec{v} < 0\end{cases}$$

Finding the Orbital Elements

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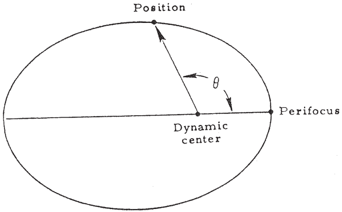
$$f = \begin{cases}f & \vec{r}\cdot\vec{v} \geq 0 \\ 360-f & \vec{r}\cdot\vec{v} < 0\end{cases}$$

- We now have finished with step 1

Propagation in Time

All orbital elements can be determined from a single observation at t_0.

- Orbital motion is periodic
  - Orbital elements allow us to predict the motion for all time.



Given a future time, t_f, we can use Kepler's equation to predict $f(t_f)$
Step 1: Use true anomaly, $f(t_0)$ to find mean anomaly, $M(t_0)$.

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right)$$

$$M = E - e\sin E$$

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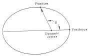

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- Don't forget the step where you find the *initial* time. This is not the time you observed the satellite, but rather the time elapsed from periapse.
- Your final time is the change in time added to the *initial* time.

Propagation in Time

Step 2: Determine mean anomaly at t_f

$$M(t_f) = M(t_0) + n(t_f - t_0)$$

Step 3: Use mean anomaly, $M(t_f)$ to find true anomaly, $f(t_f)$ using Kepler's equation.

$$M = E - e\sin E$$

$$f = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

- The true anomaly, $f(t_f)$, tells us where the satellite is at time t_f.
- But how to translate that into $\vec{r}$ and $\vec{v}$?

Propagation in Time

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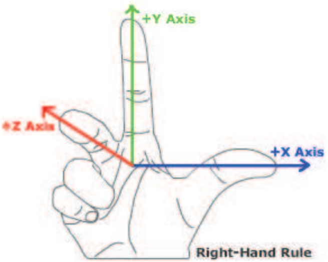
- The true anomaly, $f(t_f)$, tells us where the satellite is at time t_f.
- But how to translate that into $\vec{r}$ and $\vec{v}$?

- We assume here M is less that 2π. If not, subtract integer multiples of 2π until it is.

Coordinate Systems

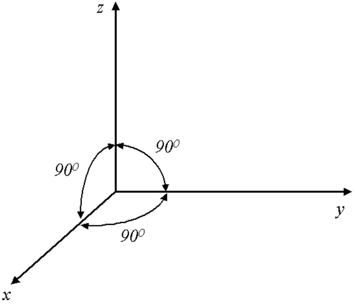
A coordinate system

- defines position variables
- defines positivity

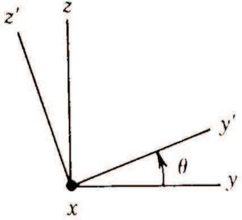


A coordinate system may be

- inertial
  - $F = ma$
- translating
- rotating



A cartesian coordinate system has right angles and is right-handed.



Example: Given a vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a rotation θ, about the y-axis,

$$\vec{v}' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \end{bmatrix}$$

The matrix is called a *rotation matrix*.

There is also a right-hand rule for Rotation.

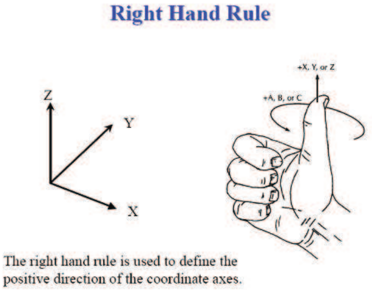
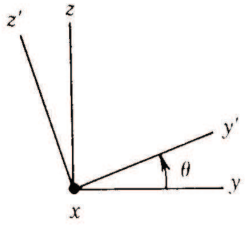


Figure: Positive Rotations

Rotation is counterclockwise when axis is pointing toward your eye.



Rotation matrices can be used to calculate the effect of **ANY** rotation.

X-Axis, ϕ **:**

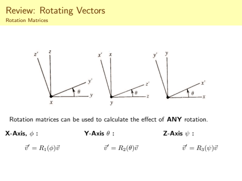
$$\vec{v}' = R_1(\phi)\vec{v}$$

Y-Axis θ **:**

$$\vec{v}' = R_2(\theta)\vec{v}$$

Z-Axis ψ **:**

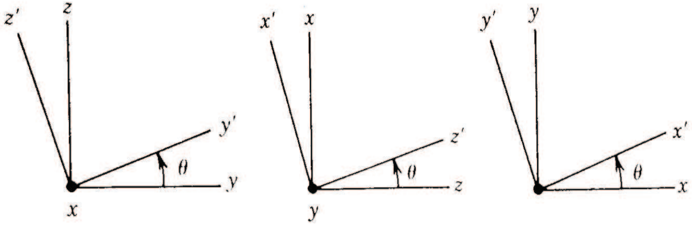
$$\vec{v}' = R_3(\psi)\vec{v}$$



- Technically, any unitary matrix is a rotation matrix. That is, given a unitary matrix, we can identify an axis of rotation and a rotation angle.
- The rotation matrices we list on the next slide correspond to rotations about the principle axes.

Review: Rotating Vectors

Rotation Matrices



The rotation matrices are (for reference):

Roll (X-Axis) (ϕ) :

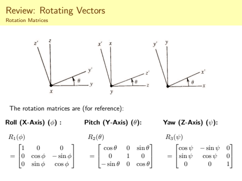
$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Pitch (Y-Axis) (θ):

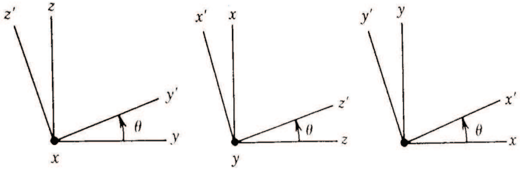
$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Yaw (Z-Axis) (ψ):

$$R_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- Note that these rotation matrices are also used for converting between reference frames. However, conversion is a bit trickier as conversion to a positively rotated reference frame actually involves a *negative* rotation of the vector.
- In this lecture, we are actually primarily interested in converting between coordinate systems (ECI$\rightarrow$perifocal$\rightarrow$satellite normal), so our angles will tend to be negative.
- That is, we will express $\vec{r}$ and $\vec{v}$ in the satellite normal (SN) coordinates and then convert to ECI.
- Our approach is a bit backward, however, because it is easier to visualize the rotations from ECI$\rightarrow$SN than the rotations from SN$\rightarrow$ECI.



Rotation matrices, can be used to calculate a sequence of rotations:

Roll-Pitch-Yaw:

$$\vec{v}_{RPY} = R_3(\psi)R_2(\theta)R_1(\phi)\vec{v}$$

Note the *order* of multiplication is critical.

$$\vec{v}_{RPY} = \left(R_3(\psi)\left(R_2(\theta)\left(R_1(\phi)\vec{v}\right)_1\right)_2\right)_3$$

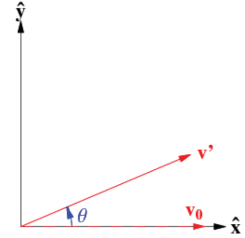
Review: Coordinate Rotations

Coordinate rotations are different than vector rotations

Case 1: Rotation of a vector in a fixed coordinate system.

Consider rotation of $\vec{r}$ around the $\hat{x}$ axis by θ and around the $\hat{z}$ axis by ω

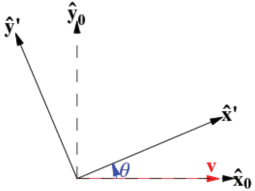
$$\vec{r}' = R_3(\omega)R_1(\theta)\vec{r}$$

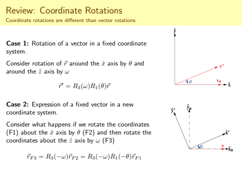


Case 2: Expression of a fixed vector in a new coordinate system.

Consider what happens if we rotate the coordinates (F1) about the $\hat{x}$ axis by θ (F2) and then rotate the coordinates about the $\hat{z}$ axis by ω (F3)

$$\vec{r}_{F3} = R_3(-\omega)\vec{r}_{F2} = R_3(-\omega)R_1(-\theta)\vec{r}_{F1}$$





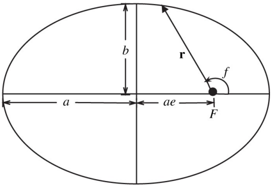
- **To be absolutely clear:** If we obtain the principle axes of F2 by rotating the principle axes of F1 through rotation matrix $R(\theta)$, then if $\vec{r}_{F1}$ is a vector expressed in F1, then $\vec{r}_{F2} = R(-\theta)\vec{r}_{F1}$ is the same vector expressed in coordinates $F2$.

- A common transformation is position in ECI to position in ECEF. These coordinates differ only by a rotation about $\hat{z}$ equal to the Greenwich sidereal time.
- Local sidereal time is given by the hour angle of the FPOA at that time and place.
- A sidereal day is 23 h, 56 min, 4s
- Rotation of a position vector from ECI to ECEF uses $R_3(-\theta_{GST})$

Note: Our method is slightly different than the book. You are free to take either approach.

Perifocal Coordinates (PQ):

- $\hat{x} = \vec{e}/e$.
- $\hat{z} = \vec{h}/h$
- $\hat{y}$ by RHR



Position in perifocal (PQ) frame is simple.

$$\vec{r}_{PQW} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \text{where} \qquad r = \frac{p}{1+e\cos f}$$

Finding $\vec{r}$ and $\vec{v}$

The Perifocal Frame

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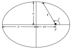

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- Actually, this is a rotation. The position vector in the SN frame is

$$\vec{r}_{SN} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$$

- We obtain the SN frame from the PQ frame using a positive rotation about $\hat{z}$ by angle f. Hence

$$\vec{r}_{SN} = R_3(-f)\vec{r}_{PQW}$$

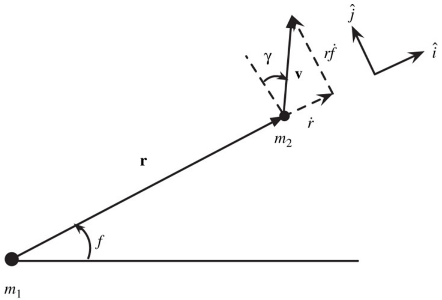
- Or, conversely,

$$\vec{r}_{PQW} = R_3(f)\vec{r}_{SN}$$

Velocity in the Perifocal Frame

Recall our original expression for 2D velocity in satellite normal frame.

$$\vec{v}_{SN} = \dot{r}\hat{i} + r\dot{f}\hat{j} = \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix}$$



- To get to the perifocal frame we rotate backwards by angle f.
- Can use rotation matrix $R_3(f)$.

$$\vec{v}_{PQW} = R_3(f) \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{r}\cos f - r\dot{f}\sin f \\ \dot{r}\sin f + r\dot{f}\cos f \\ 0 \end{bmatrix}$$

Velocity in the Perifocal Frame

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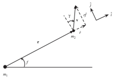

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- As before, we obtain the SN frame from the PQ frame using a positive rotation about $\hat{z}$ by angle f. Hence

$$\vec{v}_{PQW} = R_3(f)\vec{v}_{SN}$$

- PQW are the unit vectors traditionally associated with perifocal.

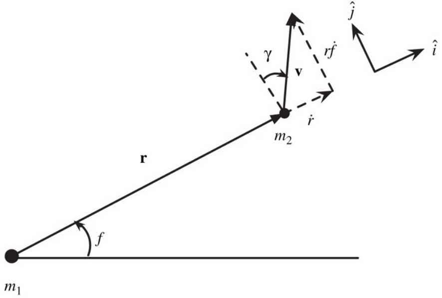
Velocity in the Perifocal Frame

Now recall $h = r^2 \dot{f}$. Hence we can simplify

$$r\dot{f} = \sqrt{\frac{\mu}{p}}\,(1 + e\cos f)\,.$$

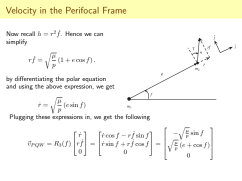
by differentiating the polar equation and using the above expression, we get

$$\dot{r} = \sqrt{\frac{\mu}{p}}\,(e\sin f)$$



Plugging these expressions in, we get the following

$$\vec{v}_{PQW} = R_3(f)\begin{bmatrix}\dot{r}\\ r\dot{f}\\ 0\end{bmatrix} = \begin{bmatrix}\dot{r}\cos f - r\dot{f}\sin f\\ \dot{r}\sin f + r\dot{f}\cos f\\ 0\end{bmatrix} = \begin{bmatrix}-\sqrt{\frac{\mu}{p}}\sin f\\ \sqrt{\frac{\mu}{p}}\,(e+\cos f)\\ 0\end{bmatrix}$$



Derivation: We use

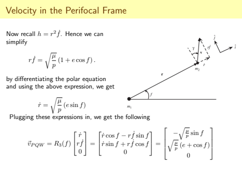
$$r = \frac{p}{1 + e\cos f} \qquad h = r^2 \dot{f}$$

For the first term,

$$r\dot{f} = \frac{h}{r} = h\frac{1 + e\cos f}{p} = \frac{h}{p}(1 + e\cos f) = \frac{\sqrt{\mu p}}{p}(1 + e\cos f) = \sqrt{\frac{\mu}{p}}(1 + e\cos f).$$

For the second term (recall $r^2\dot{f} = h$):

$$\begin{aligned}
\dot{r} &= -\frac{p}{(1 + e\cos f)^2} e\sin f\,\dot{f} \\
&= \frac{p^2}{(1 + e\cos f)^2}\frac{e}{p}\sin f\,\dot{f} \\
&= r^2\frac{e}{p}\sin f\,\dot{f} \\
&= \frac{h}{p}e\sin f = \sqrt{\frac{\mu}{p}}e\sin f
\end{aligned}$$



Derivation of the velocity term,

$$r\dot{f} = \sqrt{\frac{\mu}{p}}(1 + e\cos f) \qquad\qquad \dot{r} = \sqrt{\frac{\mu}{p}}e\sin f$$

$$\begin{aligned}
\vec{v}_{PQW} &= \begin{bmatrix}\dot{r}\cos f - r\dot{f}\sin f\\ \dot{r}\sin f + r\dot{f}\cos f\\ 0\end{bmatrix}\\
&= \begin{bmatrix}\sqrt{\frac{\mu}{p}}e\sin f\cos f - \sqrt{\frac{\mu}{p}}(\sin f + e\cos f\sin f)\\ \sqrt{\frac{\mu}{p}}e\sin^2 f + \sqrt{\frac{\mu}{p}}(\cos f + e\cos^2 f)\\ 0\end{bmatrix}\\
&= \begin{bmatrix}-\sqrt{\frac{\mu}{p}}\sin f\\ \sqrt{\frac{\mu}{p}}e + \sqrt{\frac{\mu}{p}}\cos f\\ 0\end{bmatrix} = \begin{bmatrix}-\sqrt{\frac{\mu}{p}}\sin f\\ \sqrt{\frac{\mu}{p}}(e + \cos f)\\ 0\end{bmatrix}
\end{aligned}$$

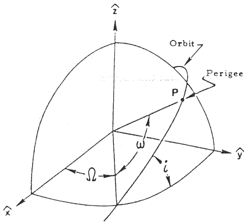
Coordinate Rotations

Perifocal to ECI Transformation

$$\vec{r}_{PQW} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}}\sin f \\ \sqrt{\frac{\mu}{p}}\left(e+\cos f\right) \\ 0 \end{bmatrix}$$

The Perifocal coordinates can be reached from ECI via 3 rotations.

1. Rotate Ω about $\hat{z}$
2. Rotate i about $\hat{x}$
3. Rotate ω about $\hat{z}$



As mentioned, rotating coordinates has the *opposite* effect of rotating the vector. Thus a vector $\vec{r}_{ECI}$ in ECI coordinates can be expressed as

$$\vec{r}_{PQW} = R_3(-\omega)R_1(-i)R_3(-\Omega)\vec{r}_{ECI}$$

Coordinate Rotations

Perifocal to ECI Transformation

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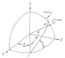

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$$\vec{r}_{PQW} = R_3(-\omega)R_1(-i)R_3(-\Omega)\vec{r}_{ECI}$$

- To rotate the x-y-z ECI coordinate system into the perifocal: Rotate z-axis by Ω. This aligns the x-axis with the line of nodes. Rotate by angle i about the line of nodes (x-axis). The plane is now correct, but the eccentricity vector is aligned with the line of nodes. Rotate about the z-axis by angle ω to correctly place the eccentricity vector.
- The sequence of rotations from ECI to PQW is Ω, i, ω.

Perifocal to ECI Transformation

Thus to convert a PQW vector to ECI, we can

$$\vec{r}_{ECI} = R_3(\Omega)R_1(i)R_3(\omega)\vec{r}_{PQW} = R_{PQW\to ECI}\vec{r}_{PQW}$$

$$R_{PQW\to ECI} = \begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\omega & -\sin\omega & 0\\ \sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i\\ \sin\omega\sin i & \cos\omega\sin i & \cos i\end{bmatrix}$$

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- Recall for rotation matrices,

$$R(\theta)^{-1} = R(\theta)^T = R(-\theta)$$

$\vec{r}$ and $\vec{v}$ in ECI

Finally, we can express our $\vec{r}$ and $\vec{v}$ in ECI.

$$\vec{r}_{ECI} = R_{PQW \to ECI} \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \vec{v}_{ECI} = R_{PQW \to ECI} \begin{bmatrix} -\sqrt{\frac{\mu}{p}}\sin f \\ \sqrt{\frac{\mu}{p}}\left(e + \cos f\right) \\ 0 \end{bmatrix}$$

Matrix multiplication is not hard:

$$\vec{r} = \begin{bmatrix} r(\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos i) \\ r(\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos i) \\ r\sin(\omega+f)\sin i \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -\frac{\mu}{h}\left(\cos\Omega\left(\sin(\omega+f) + e\sin\omega\right) + \sin\Omega\left(\cos(\omega+f) + e\cos\omega\right)\cos i\right) \\ -\frac{\mu}{h}\left(\sin\Omega\left(\sin(\omega+f) + e\sin\omega\right) - \cos\Omega\left(\cos(\omega+f) + e\cos\omega\right)\cos i\right) \\ \frac{\mu}{h}\left(\cos(\omega+f) + e\cos\omega\right)\sin i \end{bmatrix}$$

Numerical Example: $\vec{r}$ and $\vec{v}$ in ECI

Problem: Given the following orbital elements, find $\vec{r}$ and $\vec{v}$.

$$a = 35,960km = 5.64ER \qquad e = .832 \qquad f = 92.335\deg$$
$$i = 87.87\deg \qquad \Omega = 227.9\deg \qquad \omega = 53.39\deg$$

Solution: First solve for r and h.

$$p = a(1-e^2) = 1.735ER$$

$$r = \frac{p}{1+e\cos f} = 1.7947$$

$p = h^2/\mu$, so

$$h = \sqrt{p} = 1.3172.$$

Now in perifocal coordinates

$$\vec{r}_{PQW} = \begin{bmatrix} -.07319 \\ 1.7947 \\ 0 \end{bmatrix} \qquad \vec{v}_{PQW} = \begin{bmatrix} -.7585 \\ .6013 \\ 0 \end{bmatrix}$$

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- Note that $\mu = 1$ in this case because we are in universal coordinates.

Numerical Example: $\vec{r}$ and $\vec{v}$ in ECI

We can find the rotation matrices in Matlab using the following commands:

```
R3w = [cosd(w) -sind(w) 0; sind(w) cosd(w) 0; 0 0 1 ];
R1 = [1 0 0; 0 cosd(i) -sind(i) ; 0 sind(i) cosd(i)];
R3Om = [cosd(Om) -sind(Om) 0; sind(Om) cosd(Om) 0; 0 0 1 ];
```

Then compute the position and velocity vectors:

```
rECI = R3O*R1*R3w*rPQW
vECI = R3O*R1*R3w*vPQW
```

which yields

$$\vec{r}_{ECI} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER \qquad \vec{v}_{ECI} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix} ER/TU$$

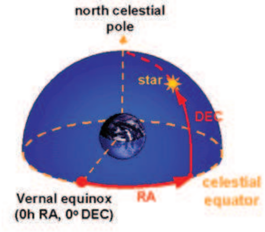
Of course we could have simply used the formulae.

Question: Now that we have $\vec{r}$ and $\vec{v}$ in ECI, what do we do with them?

Answer:

- Tracking
- Communication
- Interception
- Astronomy

For all of these applications, we need to know where to look.



1. The sky is big.
2. Satellites are small.

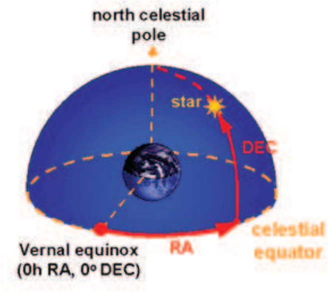
To track a satellite or star, the position vector must be translated into a direction.

These directions are declination and right ascension.

Right Ascension

Definition 1.

Right Ascension, α is the angle the position vector makes with the FPOA when projected onto the reference plane.



Initially suppose we are at the center of the earth. If $\vec{r} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$, the projection is simply $\begin{bmatrix} r_1 & r_2 \end{bmatrix}$. Thus

$$\tan(\alpha) = \frac{r_2}{r_1}$$

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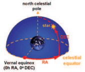

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$$\tan(\alpha) = \frac{r_2}{r_1}$$

- If we are not at the center of the earth, then the vector $\vec{r}$ is relative to our current position, $\vec{r}_{site}$, so

$$\vec{r} = \vec{r}_{rel} = \vec{r}_{sat} - \vec{r}_{site}$$

- all of these vectors are in the ECI frame

To calculate $\vec{r}_{site}$:

$$\vec{r}_{site} = R_3(\theta_{LST}) R_2(\phi) \begin{bmatrix} R_e \\ 0 \\ 0 \end{bmatrix}$$

where

- R_e is the radius of the Earth.
- θ_{LST} is Local Sidereal Time. $\theta_{LST} = \theta + \theta_{GMT}$ (θ is East longitude)
- ϕ is your latitude.

FYI: your position vector in ECI is

$$\vec{r}_{site} = R_3(\theta_{LST}) R_2(\phi) \begin{bmatrix} R_e \\ 0 \\ 0 \end{bmatrix} = R_e \begin{bmatrix} \cos\phi \cos\theta_{LST} \\ \cos\phi \sin\theta_{LST} \\ \sin\phi \end{bmatrix}$$

Declination

Definition 2.

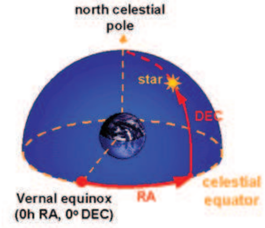
Declination, δ is the angle the position vector makes with the reference plane.

Again, simple geometry yields

$$\sin\delta = \frac{r_3}{r}$$

or

$$\tan\delta = \frac{r_3}{\sqrt{r_1^2 + r_2^2}}$$

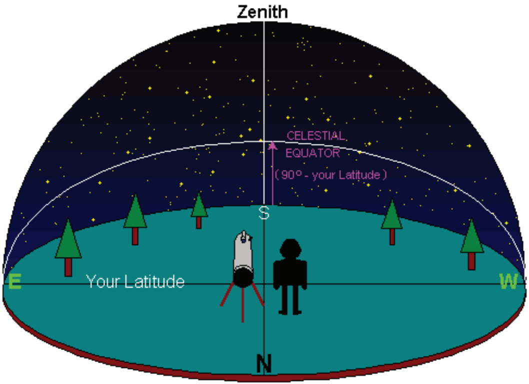


For a point on the surface of the earth, the observer must use

$$\vec{r}_{rel} = \vec{r}_{sat} - \vec{r}_{site}$$

to calculate the right ascension and declination.
Question: How to find Jupiter?

Observation using α and δ

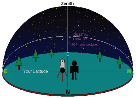


The Celestial Equator is up (90°- your Latitude) from the S horizon

Step 1: Locate the Equatorial plane.

- When facing due south, the equatorial plane will be at $90^\circ - \phi$, where ϕ is your latitude.

Observation using α and δ



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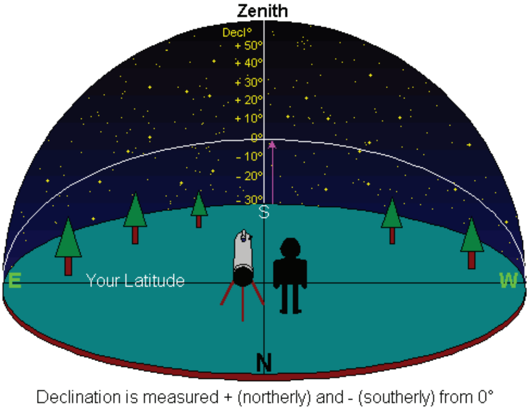
- When facing due south, the equatorial plane will be at $90° - \phi$, where ϕ is your latitude.

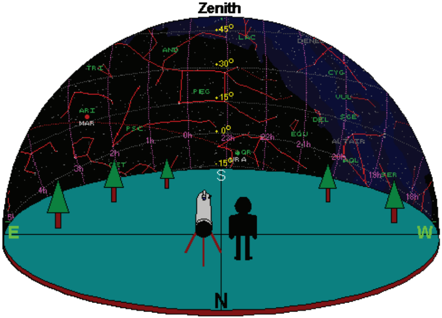
- Phoenix latitude is $\phi = 33.45°$ N
- At the equator, equatorial plane is directly overhead.
- At the north pole, equatorial plane is at the horizon.
- Note that here we are orienting here using the ECI coordinates by locating the principal axes of the ECI coordinates (north and FPOA), as opposed to rotating the position vector into ECEF, then local horizontal coordinates.

Observation using α and δ



Step 2: From the Equatorial plane, measure up/down to declination line.

Observation using α and δ

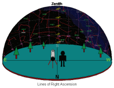


Lines of Right Ascension

Step 3: Determine right ascension, α_S of due south.

- This is given by Local Sidereal Time.
- Consult a table or do the conversion (not covered here).
- There *is* an app for that.
  - Local Sidereal Time, LSTclock, and Skyfari for iphone

Observation using α and δ



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- Local sidereal time is the angle between the FPOA and the local meridian.
- Take GMT and add the difference in longitude (positive for east)

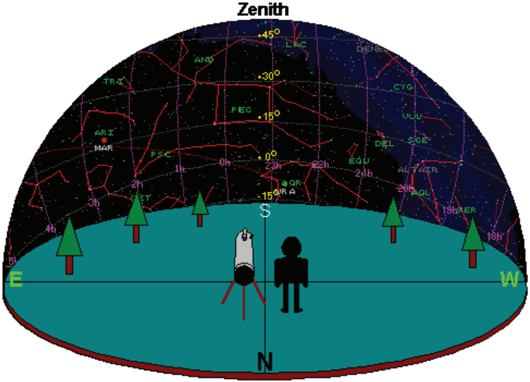
$$\theta_{LST} = \theta_{GST} + \lambda$$

- The longitude for Phoenix is 112° W

$$\lambda = 360^\circ - 112^\circ$$

- But, of course, you need to know θ_{GST}

Observation using α and δ



Lines of Right Ascension

Step 4: Find desired α_{RA} relative to α_S.

- Measure $\alpha_{RA} - \theta_{LST}$ degrees to the left of due south.

Observation using α and δ

RA/Dec Coordinates

RA/Dec Coordinates

Problem: Suppose we are in a spacecraft in the following orbit

$$a = 60,000km = 9.41ER \qquad e = .9 \qquad f = 130^\circ$$
$$i = 80^\circ \qquad \Omega = 220^\circ \qquad \omega = 70^\circ$$

We would like to use our laser cannon to destroy a defense satellite/Ballistic Missile in the following orbit.

$$a = 35,960km = 5.64ER \qquad e = .832 \qquad f = 92.335^\circ$$
$$i = 87.87^\circ \qquad \Omega = 227.9^\circ \qquad \omega = 53.39^\circ$$

What range, Right Ascension and declination should we give to the targeting computer?

Step 1: Find our position vector. We use the same Matlab script as before.

$$\vec{r}_1 = \begin{bmatrix} 4.71 \\ 5.97 \\ -8.74 \end{bmatrix} ER$$

Numerical Example

Step 2: The position vector

$$\vec{r}_2 = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER$$

Step 3: The relative position vector

$$\vec{r}_2 - \vec{r}_1 = \begin{bmatrix} -3.7 \\ -4.89 \\ 9.75 \end{bmatrix}$$

Step 4: Translate into RA and declination. Use Matlab commands

```
dec=atan2(rrel(3),sqrt(rrel(1)^2 + rrel(2)^2))
RA=atan2(rrel(2),rrel(1))
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Yields $\delta = 1.0097 rad$, $\alpha = -2.2173 rad$.

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Yields $\delta = 1.0097 rad$, $\alpha = -2.2173 rad$.

- Dont forget to adjust your declination for latitude and RA for θ_{LST}
- It may be hard to locate the equatorial or ecliptic planes in space (use star tracker, sun tracker, earth tracker).

Summary

This Lecture you have learned:

How to convert between

- a, e, i, Ω, ω, f
- $\vec{r}$ and $\vec{v}$

How to translate $\vec{r}$ and $\vec{v}$ into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Next Lecture: Transfer Orbits