Spacecraft Dynamics and Control

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Lecture 8: Impulsive Orbital Maneuvers
In this Lecture, you will learn:

Coplanar Orbital Maneuvers
- Impulsive Maneuvers
  - $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

**Numerical Problem:** Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and $\Delta v$).
Changing Orbits

Suppose we have designed our ideal orbit.

- We have chosen $a_d, e_d, i_d, \Omega_d, \omega_d$
- We are currently in orbit $a_0, e_0, i_0, \Omega_0, \omega_0$
  
  ▶ Determined from current position $\vec{r}$ and velocity $\vec{v}$.

**Question:**

- How to get from current orbit to desired orbit?
- What tools can we use?
- What are the constraints?

Unchanged, the object will remain in initial orbit indefinitely.
Changing Orbits

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For now, we don’t care about $f$ (time)
  - Lambert’s Problem
  - Can correct using phasing

Don’t care about efficiency

true anomaly ($f$) determines phasing within the orbit and is easily altered post-insertion.
How to create a $\Delta v$

$\Delta v$ is our tool for changing orbits

Velocity change is caused by thrust.
- For constant thrust, $F$,

$$v(t) = v(0) + \frac{F}{m} \Delta t$$

- for a desired $\Delta v$, the time needed is

$$\Delta t = \frac{m \Delta v}{F}$$

We assume $\Delta t$ and $\Delta \vec{r}$ are negligible for a $\Delta v$.
- No continuous thrust transfers
- Although these are increasingly important.

The change in position is

$$\Delta \vec{r}(t) = \frac{m \Delta v^2}{2F}$$
How to create a $\Delta v$

- For fixed $\Delta v$, if $\frac{m}{F}$ is small, the $\Delta r^2$ is small
- We will assume $\Delta r^2 = 0$

$$v(t) = v(0) + \frac{F}{m} t$$

so

$$t = \Delta v \frac{m}{F}$$

Now,

$$r(t) = r(0) + v(0)t + \frac{F}{2m} t^2$$

$$\Delta r = v(0)t + \frac{F}{2m} \Delta v^2 \frac{m^2}{F^2}$$

$$= v(0) \Delta v \frac{m}{F} + \frac{\Delta v^2}{2} \frac{m}{F} = \left( v(0) \Delta v + \frac{\Delta v^2}{2} \right) \frac{m}{F}$$

However, we can ignore the $v(0)$ if we are considering deviation from a nominal path.
\( \Delta V \) moves the vacant focus of the orbit

Orbit maneuvers are made through changes in velocity.

- \( \vec{r} \) and \( \vec{v} \) determine orbital elements.
- Our first constraint is \textit{continuity}.
  - New orbit must also pass through \( \vec{r} \).
  - Cannot jump from one orbit to another instantly
  - If the current and target orbit don’t intersect, a \textit{transfer orbit} is required.
Equations involving velocity

\[ v_c = \sqrt{\frac{\mu}{r_c}} \quad \text{circular orbit} \]

\[ v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)} \quad \text{vis-viva} \]

\[ v_p = \sqrt{\frac{\mu}{a} \left( \frac{1+e}{1-e} \right)} \quad \text{periapse velocity} \]

\[ v_a = \sqrt{\frac{\mu}{a} \left( \frac{1-e}{1+e} \right)} \quad \text{apoapse velocity} \]

\[ v_{esc} = \sqrt{\frac{2\mu}{r}} \quad \text{escape velocity} \]

\[ \vec{v} = \begin{bmatrix} -\frac{\mu}{h} \left( \cos \Omega \left( \sin(\omega + f) + e \sin \omega \right) + \sin \Omega \left( \cos(\omega + f) + e \cos \omega \right) \cos i \right) \\ -\frac{\mu}{h} \left( \sin \Omega \left( \sin(\omega + f) + e \sin \omega \right) - \cos \Omega \left( \cos(\omega + f) + e \cos \omega \right) \cos i \right) \\ \frac{\mu}{h} \left( \cos(\omega + f) + e \cos \omega \right) \sin i \end{bmatrix} \]
What can we do with a $\Delta v$ Maneuver?

$\Delta v$ refers to the difference between the initial and final velocity vectors.

A $\Delta v$ maneuver can:

- Raise/lower the apogee/perigee
- A change in inclination
- Escape
- Reduction/Increase in period
- Change in RAAN
- Begin a 2+ maneuver sequence of burns.
  - Creates a **Transfer Orbit**.

We’ll start by talking about coplanar maneuvers.
What can we do with a $\Delta v$ Maneuver?

- Raise/lower the apogee/perigee is performed at perigee/apogee
- A change in inclination is *usually* performed at the equatorial plane (any inclination achievable from this point).
- Small changes in period help with phase changes $f(t)$.
- Change in RAAN should be done as far from equatorial plane as possible.
Definition 1.

Coplanar Maneuvers are those which do not alter $i$ or $\Omega$.

Example: Simple Tangential Burn

- For maximum efficiency, a burn must occur at $0^\circ$ flight path angle
  - $\dot{\gamma} = 0$
- Tangential burns can occur at perigee and apogee
Single Burn Coplanar Maneuvers

Apogee or Perigee raising or lowering.

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Example: Simple Tangential Burn
- For maximum efficiency, a burn must occur at 0° flight path angle \( \dot{\gamma} = 0 \).
- Tangential burns can occur at perigee and apogee.

We will explain why we want \( \angle FPA = 0^\circ \) in Lecture 9, when we discuss the Oberth effect.
Example: Insertion into a Parking Orbit
A perigee raising maneuver

Suppose we launch from the surface of the earth.
- This creates an initial elliptic orbit which will re-enter.
- To circularize the orbit, we plan on using a burn at apogee.

Problem: We are given $a$ and $e$ of the initial elliptic orbit. Calculate the $\Delta v$ required at apogee to circularize the orbit.
Example: Insertion into a Parking Orbit

A perigee raising maneuver

Calculating the $\Delta v$: To raise the perigee, we burn at apogee. At apogee, we have that

$$r_{a_0} = a_0(1 + e_0)$$

From the vis-viva equation, we can calculate the velocity at apogee.

$$v_{a_0} = \sqrt{\mu \left( \frac{2}{r_{a_0}} - \frac{1}{a_0} \right)} = \sqrt{\frac{\mu}{a_0} \left( \frac{1 - e_0}{1 + e_0} \right)}$$

Our target orbit is circular with radius $r_d = a_d = r_{a_0}$. The velocity of the target orbit is constant at

$$v_c = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1 + e)}}$$

Therefore, the $\Delta v$ required to circularize the orbit is

$$\Delta v = v_c - v_{a_0} = \sqrt{\frac{\mu}{a_0(1 + e_0)}} - \sqrt{\frac{\mu}{a_0} \left( \frac{1 - e_0}{1 + e_0} \right)}$$

- It is unusual to launch directly into the desired orbit. Instead we use the parking orbit while waiting for more complicated orbital maneuvers.
Given a Desired Transfer Orbit
How to calculate the $\Delta v$'s?

Let's generalize the parking orbit example to the case of a transfer orbit.

**Definition 2.**

- **The Initial Orbit** is the orbit we want to leave.
- **The Target Orbit** is the orbit we want to achieve.
- **The Transfer Orbit** is an orbit which intersects both the initial orbit and target orbit.

**Step 1:** Design a transfer orbit - $a, e, i, \omega, \Omega, f$

**Step 2:** Calculate $\vec{v}_{tr,1}$ at the point of intersection with initial orbit.

**Step 3:** Calculate initial burn to maneuver into transfer orbit.

$$\Delta v_1 = \vec{v}_{tr,1} - \vec{v}_{init}$$
Given a Desired Transfer Orbit

- Spacecraft Dynamics

How to calculate the \( \Delta v \)'s?

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**Step 1:** Design a transfer orbit - \( a, e, i, \omega, \Omega, f \)
**Step 2:** Calculate \( \vec{v}_{tr,1} \) at the point of intersection with initial orbit.
**Step 3:** Calculate initial burn to maneuver into transfer orbit.
\[
\Delta v_1 = \vec{v}_{tr,1} - \vec{v}_{ini}
\]

- Note that in the illustration, the transfer orbit is not a Hohman transfer, which is the most common type of transfer orbit.
- **Step 1** may be **VERY HARD** because you may not know what \( f_{tr} \) will be in your transfer orbit!
Step 4: Calculate $\vec{v}_{tr,2}$ at the point of intersection with target orbit.

Step 5: Calculate velocity of the target orbit, $\vec{v}_{fin}$, at the point of intersection with transfer orbit.

Step 6: Calculate the final burn to maneuver into target orbit.

$$\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}$$
Step 4 may be easy if you can find the point of intersection, $\vec{r}_{tr,2}$, since you can then use the polar equation to find $f_{tr,2}$. However, finding the point of intersection may be hard.
The Choice of Transfer Orbit

Constraints:

- The transfer orbit must intersect both current and target orbit
- The $\Delta v$'s for entering transfer and orbital insertion are limited by $\Delta v$ budget
  - Typically limits us to elliptic transfers.
- There may be constraints on elapsed time.
The Choice of Transfer Orbit

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The image is of an unproven conjecture that the most efficient 2-burn transfer between 2 coplanar orbits always uses a tangential burn.

- Feel free to find a counterexample!
- Use a brute force search approach using Lambert’s problem to calculate $\Delta v$'s.
The Choice of Transfer Orbit

Continuity Constraints affect range of $a$ and $e$

There are many orbits which intersect both the initial and target orbits. However, there are some constraints.

Consider

- Circular initial orbit of radius $r_1$
- Circular target orbit of radius $r_2 > r_1$

Obviously, the transfer orbit must satisfy

$$r_p = \frac{p}{1 + e} \leq r_1$$

and

$$r_a = \frac{p}{1 - e} \geq r_2$$
The Choice of Transfer Orbit

Continuity Constraints affect range of $a$ and $e$

There are many orbits which intersect both the initial and target orbits. However, there are some constraints.

Consider

- Circular initial orbit of radius $r_1$
- Circular target orbit of radius $r_2 > r_1$

Obviously, the transfer orbit must satisfy

- The $r_p$ constraint says the transfer orbit must intersect the initial orbit.
- The $r_a$ constraint says the transfer orbit must intersect the target orbit.
- The plot illustrates the range of realizable $p$ and $e$ for given initial and target radii.
- The lines represent

\[ p \geq r_2(1 - e) \quad \iff \quad e > 1 - \frac{p}{r_2} \]

and

\[ p < r_1(1 + e) \quad \iff \quad e > \frac{p}{r_1} - 1 \]

- Again, we assume no constraint on timing or phasing.
Transfer Orbits in Fixed Time

Constraints on Transfer Time

Occasionally, we want to arrive at

- A certain point in the target orbit, $\vec{r}_2$
- at a certain time, $t_f$

Finding the necessary transfer orbit is **Lambert’s Problem**.

Primary Applications are:

- Targeting
- Rendez-vous

We will come back to the section on Lambert’s problem.
Occasionally, we want to arrive at
• A certain point in the target orbit, \( \vec{r}_2 \)
• at a certain time, \( t_f \)
Finding the necessary transfer orbit is Lambert’s Problem.

Primary Applications are:
• Targeting
• Rendez-vous

We will come back to the section on Lambert’s problem.

The plot shows 2 possible transfer orbits between point \( P_1 \) and \( P_2 \).
Constraints on $\Delta v$ budget

What is a minimum energy transfer orbit?

The critical resource in space travel is $\Delta v$.

- The $\Delta v$ budget is fixed at takeoff.
- Refueling is not usually possible.
- If you run out of $\Delta v$, bad things happen.

$\Delta v$ can increase or decrease the energy of an orbit.

- The energy difference between 2 orbits must come from somewhere.

$$\Delta E_{\text{min}} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1}$$

- The closer $E_{\text{cost}}$ is to $E_{\text{min}}$, the more efficient the transfer
- $\Delta v$ does not translate directly to Energy changes, however.
- More on this effect later
The energy GAIN for each $\Delta v$ is actually larger - depending on the initial velocity. We will discuss this more carefully next lecture.

Note energy is NOT conserved here, so $\frac{\Delta v^2}{2} \neq \Delta E_{\text{min}}$. 

Constraints on $\Delta v$ budget
The Hohmann Transfer
A Minimum Energy Orbit?

The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.

- Proposed by Hohmann (1925)
  - Why?
- Proven for circular target orbits by Lawden (1952)
- Proven for coaxial elliptical initial and target orbits by Thompson (1986)
The Hohmann Transfer

A Minimum Energy Orbit?

The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.

- Proposed by Hohmann (1925)
- Proven for circular target orbits by Lawden (1952)
- Proven for coaxial elliptical initial and target orbits by Thompson (1986)

- Also first proposed use of separable lunar landers
- Did not participate in Nazi rocket program.
- Died of hunger/stress after allied bombardment of Essen
- Optimality was originally a conjecture.

- Published in “Die Erreichbarkeit der Himmelskörper (The Attainability of Celestial Bodies)” (1925) [PDF Available Here]
The Hohmann Transfer

We will first consider the circular case.

Theorem 3 (The Hohmann Conjecture).

The energy-optimal transfer orbit between two circular orbits of radii $r_1$ and $r_2$ is an elliptic orbit with

$$r_p = r_1 \quad \text{and} \quad r_a = r_2$$

This yields the orbital elements of the transfer orbit $(a, e)$ as

$$a = \frac{r_a + r_p}{2} = \frac{r_1 + r_2}{2} \quad \text{and} \quad e = 1 - \frac{r_p}{a} = \frac{r_2 - r_1}{r_2 + r_1}$$
The Hohmann Transfer

To calculate the required $\Delta v_1$ and $\Delta v_2$, the initial velocity is the velocity of a circular orbit of radius $r_1$

$$v_{init} = \sqrt{\frac{\mu}{r_1}}$$

The required initial velocity is that of the transfer orbit at perigee. From the vis-viva equation,

$$v_{trans,p} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = \sqrt{2\mu} \sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} = \sqrt{2\mu \frac{r_2}{r_1(r_1 + r_2)}}$$

So the initial $\Delta v_1$ is

$$\Delta v_1 = v_{trans,p} - v_{init} = \sqrt{2\mu \frac{r_2}{r_1(r_1 + r_2)}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right)$$

The velocity of the transfer orbit at apogee is

$$v_{trans,a} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = \sqrt{2\mu \frac{r_1}{r_2(r_1 + r_2)}}$$
The Hohmann Transfer

The required velocity for a circular orbit at apogee is

\[ v_{\text{fin}} = \sqrt{\frac{\mu}{r_2}} \]

So the final \( \Delta v_2 \) is

\[ \Delta v_2 = v_{\text{fin}} - v_{\text{trans,a}} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2 \mu \frac{r_1}{r_2 (r_1 + r_2)}} = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2 r_1}{(r_1 + r_2)}} \right) \]

Thus we conclude to raise a circular orbit from radius \( r_1 \) to radius \( r_2 \), we use

\[ \Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2 r_2}{(r_1 + r_2)}} - 1 \right) \]

\[ \Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2 r_1}{(r_1 + r_2)}} \right) \]
Hohmann Transfer Illustration
The Hohmann Transfer

Transfer Time

The Hohmann transfer is optimal

- Only for impulsive transfers
  - Continuous Thrust is not considered
- Only for two impulsive transfers
  - A three impulse transfer can be better
  - Bi-elliptics are better

The transfer time is simply half the period of the orbit. Hence

\[
\Delta t = \frac{\tau}{2} = \pi \sqrt{\frac{a^3}{\mu}}
\]

\[
= \pi \sqrt{\frac{(r_1 + r_2)^3}{8\mu}}
\]

The Hohmann transfer is also the Maximum Time 2-impulse Transfer.

- Always a tradeoff between time and efficiency
- Bielliptic Transfers extend this tradeoff.
The Hohmann Transfer

- Transfer Time
  - The Hohmann transfer is optimal
    - Only for impulsive transfers
      - Continuous Thrust is not considered
    - Only for two impulse transfers
      - A three impulse transfer can be better
      - Bi-elliptics are better
  - The transfer time is simply half the period of the orbit. Hence
    \[ \Delta t = \frac{\pi}{\sqrt{\frac{3}{8}} \mu} = \frac{\pi}{\sqrt{\frac{r_1 + r_2}{8}} \mu} \]
  - The Hohmann transfer is also the Maximum Time 2-impulse Transfer.
  - Always a tradeoff between time and efficiency
  - Bi-elliptic Transfers extend this tradeoff.

- The slowest part of the orbit is at apogee.
- Due to Oberth effect, you want to use as much \( \Delta v \) budget as possible at low altitude. Bi-elliptics use this to further reduce \( \Delta v \) at apogee (Next Lecture)
- Hohmann transfer to GEO is extremely wasteful!
Numerical Example (Parking Orbit to GEO)

**Problem:** Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and $\Delta v$).

**Solution:** We will use a Hohmann transfer between circular orbits of

$r_1 = 191.35km + 1ER = 1.03ER \quad \text{and} \quad r_2 = 35781km + 1ER = 6.61ER$

The initial velocity is

$$v_i = \sqrt{\frac{\mu}{r_1}} = .985 \frac{ER}{TU}$$

The transfer ellipse has $a = \frac{r_1 + r_2}{2} = 3.82ER$. The velocity at perigee is

$$v_{trans,1} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = 1.296 \frac{ER}{TU}$$

Thus the initial $\Delta v$ is $\Delta v_1 = 1.296 - .985 = .315 \frac{ER}{TU}$. 

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Lecture 8: Spacecraft Dynamics
Numerical Example

The velocity at apogee is

\[ v_{trans,1} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = 0.202 \frac{ER}{TU} \]

However, the required velocity for a circular orbit at radius \( r_2 \) is

\[ v_f = \sqrt{\frac{\mu}{r_2}} = 0.389 \frac{ER}{TU} \]

Thus the final \( \Delta v \) is \( \Delta v_2 = 0.389 - 0.202 = 0.182 \frac{ER}{TU} \). The second \( \Delta v \) maneuver should be made at time

\[ t_{fin} = \pi \sqrt{\frac{a^3}{\mu}} = 23.45TU = 5.256hr \]

The total \( \Delta v \) budget is \( 0.497 ER/TU \).
The Elliptic Hohmann Transfer

The Hohmann transfer is also energy optimal for coaxial elliptic orbits.

The only ambiguity is whether to make the initial burn at perigee or apogee.
• Need to check both cases
• Often better to make initial burn at perigee
  ▶ Due to Oberth Effect
Summary

This Lecture you have learned:

Coplanar Orbital Maneuvers

- Impulsive Maneuvers
  ▶ $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  ▶ Elliptic
  ▶ Circular

Next Lecture: Oberth Effect, Bi-elliptics, Out-of-plane maneuvers.