Spacecraft Dynamics and Control

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Lecture 13: The Effect of a Non-Spherical Earth
In this Lecture, you will learn:

The Non-Spherical Earth
- The gravitational potential
- Expression in the R-T-N frame
- Perturbations
  - Periodic
  - Secular

Mission Planning
- Sun-Synchronous Orbits
- Frozen Orbits
- Critical Inclination
Recall The Perturbation Equations

\[ \vec{F}_{\text{disturbance}} = R \hat{e}_R + T \hat{e}_T + N \hat{e}_N \]

### Semi-major Axis

\[
\dot{a} = 2 \sqrt{\frac{a^3}{\mu(1 - e^2)}} \left[ e R \sin f + T(1 + e \cos f) \right]
\]

### Eccentricity:

\[
\dot{e} = \sqrt{\frac{a(1 - e^2)}{\mu}} \left[ R \sin f + T(\cos f + \cos E_{ecc}) \right]
\]

### Inclination:

\[
\frac{d}{dt} \dot{i} = \sqrt{\frac{a(1 - e^2)}{\mu}} \frac{N \cos(\omega + f)}{1 + e \cos f}
\]

### RAAN:

\[
\dot{\Omega} = \sqrt{\frac{a(1 - e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i(1 + e \cos f)}
\]

### Argument of Perigee:

\[
\dot{\omega} = -\dot{\Omega} \cos i + \sqrt{\frac{a(1 - e^2)}{e^2 \mu}} \left( -R \cos f + T \frac{(2 + e \cos f) \sin f}{1 + e \cos f} \right)
\]
Recall The Perturbation Equations

\[ \vec{F}_{\text{disturbance}} = \vec{R} \hat{e}_R + \vec{T} \hat{e}_T + \vec{N} \hat{e}_N \]

**Semi-major Axis**

\[ \dot{a} = \frac{a}{\sqrt{a^3}} \sqrt{\mu} (1 - e^2) \left[ e \sin f + T (1 + e \cos f) \right] \]

**Inclination**

\[ \frac{d}{df} i = \sqrt{a (1 - e^2) \frac{\mu N}{a + e \cos f}} [\cos(\omega + f)] \]

**Eccentricity**

\[ \dot{e} = \sqrt{a (1 - e^2) \mu} \left[ e \sin f + T (\cos f + \cos E_{e_2}) \right] \]

**RAAN**

\[ \dot{\Omega} = \sqrt{a (1 - e^2) \mu N} \sin(\omega + f) \sin i (1 + e \cos f) \]

**Argument of Perigee**

\[ \dot{\omega} = -\Omega \cos i + \sqrt{a (1 - e^2) e^2 \mu} \left[ -R \cos f + T (2 + e \cos f) \sin f \right] \]

Drag is only significant in LEO. Above LEO, \( J_2 \) is more important (From Gil/Montenbruck).
Recall

Satellite-Normal Coordinate System

\[ \vec{F} = N \hat{e}_N + R \hat{e}_R + T \hat{e}_T \]

**Satellite-Normal CS (R-T-N):**

- \( \hat{e}_R \) points along the earth → satellite vector.
- \( \hat{e}_N \) points in the direction of \( \vec{h} \)
- \( \hat{e}_T \) is defined by the RHR
  - \( \hat{e}_T \cdot v > 0 \).
Recall that gravity for a point mass is

\[ \vec{F} = -\mu \frac{\vec{r}}{\|\vec{r}\|^2} \]

Gravity force derives from the potential field.

\[ \vec{F} = \nabla U \]

To find \( U \), we integrate

\[ dU = -\frac{2\pi R^2 G \sigma m_2 \sin \theta}{\rho} d\theta \]

For a uniform spherical mass,

- There is symmetry about the line \( \vec{r}_{12} \).
- The point-mass approximation holds.
For a spherical earth, $dU$ is symmetric

$$dU = -\frac{2\pi R^2 G \sigma m_2 \sin \theta}{\rho} d\theta$$

The actual gravity field
- Is not precisely spherical.
- Density varies throughout the earth.

The result is a distorted potential field.

**Figure:** The geoid, 15000:1 scale
The Non-spherical Earth

A Distorted Potential Field

For a spherical earth, \( dU \) is symmetric:

\[
\frac{dU}{d\theta} = -2\pi R^2 G \sigma m \sin^2 \theta \rho d\theta
\]

The actual gravity field
• Is not precisely spherical.
• Density varies throughout the earth.
The result is a distorted potential field.

The Geoid is the surface of gravitational and centrifugal equipotential
• Describes the surface of the ocean if it covered the entire earth

Of course, for orbit perturbations, we exclude the centrifugal potential energy.
**Socrates:** So how do we derive the potential field?

**Tycho Brahe:** We measure it!!!

**Definition 1.**

**Physical Geodesy** is the study of the gravitational potential field of the earth.

**Definition 2.**

The **Geoid** is equipotential surface which coincides with the surface of the ocean.
The first measurements of the earth were made by Eratosthenes

- Third Librarian of Library of Alexandria (240BC).
- Invented “Geography”
- Invented Latitude and Longitude
  - The difference in angle between high noon at two points on the earth.
  - Measured using deep wells
- Measured the circumference of the earth.
- May have starved to death.
● Starved himself to death after going blind and therefore being unable to read.
Geometry of Eratosthenes
Question: So how do we measure the potential field of the earth?

LAGEOS: Laser Geodynamics Satellites

1. Precisely measure the trajectory of a satellite as it orbits the earth
2. Account for drag, third-body dynamics, etc.
3. Remaining perturbation must be causes by gravitational potential

The orbits of the LAGEOS satellites are measured precisely by laser reflection.

Note: Only measures potential along path of the orbit.

- We must observe for a long time to get comprehensive data.

\[ a = 12,278 \text{ km}, \quad i = 109.8^\circ, 52.6^\circ, \]

Launch dates: 1976, 1992
The Non-spherical Earth

- Does not measure potential field directly.
- Requires this field to be fit to the trajectory data.
The Non-spherical Earth

Measuring satellite positions from earth is inaccurate.

- Atmospheric Distortion

GRACE (2002):

1. Measure the relative position of two adjacent satellites
2. Relative motion yields gradient of the potential field
3. Allows direct reconstruction of $U(\vec{r})$.

Less fancy methods:

- Survey markers
- Altimetry
- Ocean level variation

$a = 6700 km$, $i = 90^\circ$
The Non-spherical Earth

Measuring satellite positions from earth is inaccurate.

- Atmospheric Distortion

**GRACE (2002):**
1. Measure the relative position of two adjacent satellites
2. Relative motion yields gradient of the potential field
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Less fancy methods:
- Survey markers
- Altimetry
- Ocean level variation

$a = 6700\text{km}, \ i = 90^\circ$
Data from GRACE

Ocean surface equivalent
Question: So what is $U(\vec{r})$? (Needed to compute $\vec{F} = \nabla U$)
Response: Too much data to write as a function.

In order to be useful, we match the data to a few basis functions.

Coordinates: Express position using $\phi_{gc}$, $\lambda$, $r$.

- $\phi_{gc}$ is declination from equatorial plane.
- $r$ is radius
- $\lambda$ is right ascension, measured from Greenwich meridian.

We will have a function of form

$$U(\phi_{gc}, \lambda, r)$$
Note that $U$ will be defined in ECEF coordinates.

- We will need to change to ECI and ultimately RTN coordinates in order to apply the orbit perturbation equations.
- This is one of those cases where RTN is not the natural coordinate system for the force.
The potential has the form

\[ U(\phi_{gc}, \lambda, r) = \frac{\mu}{r} + U_{\text{zonal}}(r, \phi_{gc}) + U_{\text{sectorial}}(r, \lambda) + U_{\text{tesseral}}(r, \phi_{gc}, \lambda) \]

Actually, \( U_{\text{sectorial}} \) varies with \( \phi_{gc} \), but not “harmoniously”.
The Zonal Harmonics

Zonal Harmonics: These have the form

\[ U_{\text{zonal}}(r, \phi_{gc}) = \frac{\mu}{r} \sum_{i=2}^{\infty} J_i \left( \frac{R_e}{r} \right)^i P_i(\sin \phi_{gc}) \]

- \( R_e \) is the earth radius
- \( P_i \) are the Legendre Polynomials
- The \( J_i \) are determined by the Geodesy data!

Zonal harmonics vary only with latitude.
The Zonal Harmonics

Technically, the zonal harmonics are only the $P_i(\sin \phi_{gc})$ terms where the $P_i$ are the Legendre polynomials

$$P_n(x) = P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

This is Rodrigues’ formula

- This is a bit confusing, since, e.g.
  $$P_1(\sin \phi) = \cos \phi$$
  $$P_2(\sin \phi) = 3 \cos^2 \phi$$

- What is even more confusing is some texts (e.g. Curtis) measure $\phi = 90 - \phi_{gc}$.

- Then $\sin \phi_{gc}$ becomes $\cos \phi$. 

The Sectorial Harmonics

**Sectorial Harmonics:** These have the form

\[
U_{sect}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i=2}^{\infty} \left( C_{i,sect} \cos(i\lambda) + S_{i,sect} \sin(i\lambda) \right) \left( \frac{R_e}{r} \right)^i P_i(\sin \phi_{gc})
\]

- Divides globe into slices by longitude.
- Varies with \( \phi_{gc} \), but \( P_i(\sin \phi_{gc}) \) is uniformly positive.
- The \( C_{i,sectorial} \) and \( S_{i,sectorial} \) are also determined by the Geodesy data!

Sectorial harmonics vary only with longitude.
Many texts ignore the Sectorial and Tesseral Harmonics

- The effect often appears random/hard to predict. Not much secular perturbation
- The exception to this is repeating ground tracks.

If interested, “Satellite Orbits” by Gil and Montenbruck has all the dynamics well-explained.
The Tesseral Harmonics

These have the form

\[ U_{\text{tesseral}}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i,j=2}^{\infty} \left( C_{i,j} \cos(i\lambda) + S_{i,j} \sin(i\lambda) \right) \left( \frac{R_e}{r} \right)^i P_{i,j}(\sin \phi_{gc}) \]

- Divides globe into slices by longitude and latitude.
- The \( C_{i,j} \) and \( S_{i,j} \) are also determined by the Geodesy data!
The J2 Perturbation

For simplicity, we ignore all harmonics except the first zonal harmonic.

\[
\Delta U_{J2}(r, \phi_{gc}) = -\frac{\mu}{r} J_2 \left( \frac{R_e}{r} \right)^2 \left[ \frac{3}{2} \sin^2(\phi_{gc}) - \frac{1}{2} \right]
\]

This corresponds to a single band about the equator.

- The earth is 21 km wider than it is tall.
- A flattening ratio of \( \frac{1}{300} \).
- \( J_2 = .0010826 \)
- \( J_3 = .000002532 \)
- \( J_4 = .000001620 \)
For simplicity, we ignore all harmonics except the first zonal harmonic.

\[ \Delta U_{J2}(r, \phi) = -\frac{\mu}{2} \left( \frac{R_e}{r} \right)^2 \left[ \frac{3}{2} \sin^2(\phi) - \frac{1}{2} \right] \]

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Image credit: [https://ai-solutions.com/_freeflyeruniversityguide/j2_perturbation.htm](https://ai-solutions.com/_freeflyeruniversityguide/j2_perturbation.htm)
The J2 Perturbation
Defined in the Wrong Coordinate System

\[ \Delta U_{J2}(r, \phi_{gc}) = -\frac{\mu}{r} J_2 \left( \frac{R_e}{r} \right)^2 \left[ \frac{3}{2} \sin^2(\phi_{gc}) - \frac{1}{2} \right] \]

- Expressed in the ECI Frame (same as ECEF here)
- Since \( \sin \phi_{gc} = \frac{z}{r} \),

\[ \Delta U_{J2}(r, \phi_{gc}) = -\frac{\mu}{r} \frac{J_2}{2} \left( \frac{R_e}{r} \right)^2 \left[ \frac{3z^2}{r^2} - 1 \right] \]

We now calculate the perturbation force as

\[ \vec{F} = -\frac{\partial U_{J2}}{\partial r} \hat{e}_R + \frac{\partial U_{J2}}{\partial z} \hat{e}_z \]

\[ = -\mu J_2 R_e^2 \left[ \frac{3z}{r^5} \hat{e}_z + \left( \frac{3}{2r^4} - \frac{15z^2}{2r^6} \right) \hat{e}_R \right] \]

But to use our perturbation equations, we need a force expressed in the R-T-N frame.
The J2 Perturbation
Defined in the Wrong Coordinate System

\[ \Delta U_{J2}(r, \phi_{gc}) = -\mu \frac{J_2}{(Re r)^2} \left[ \frac{3}{2} \sin^2(\phi_{gc}) - \frac{1}{2} \right] \]

- Expressed in the ECI Frame (same as ECEF here)
- Since \( \sin \phi_{gc} = \hat{z} \)

\[ \Delta U_{J2}(r, \phi_{gc}) = -\mu \frac{J_2}{(Re r)^2} \left[ \frac{3}{2} \hat{z}^2 \right] \]

We now calculate the perturbation force as

\[ \vec{F} = -\partial U_{J2}/\partial r \hat{e}_R + \partial U_{J2}/\partial z \hat{e}_z \]

\[ = -\mu J_2 \frac{1}{2} \left[ \frac{3}{2} \hat{z}^2 + \frac{1}{2} \hat{e}_R - \frac{15}{4} \hat{e}_z \right] \]

But to use our perturbation equations, we need a force expressed in the R-T-N frame.

These calculations are from the 1993 version of Prussing and Conway.
Recall: Perifocal to ECI Transformation

To convert a PQW vector to ECI, we use

\[ \vec{r}_{ECI} = R_3(\Omega)R_1(i)R_3(\omega)\vec{r}_{PQW} = R_{PQW\rightarrow ECI}\vec{r}_{PQW} \]

\[
R_{PQW\rightarrow ECI} = \begin{bmatrix}
\cos \Omega & -\sin \Omega & 0 \\
\sin \Omega & \cos \Omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos i & -\sin i \\
0 & \sin i & \cos i
\end{bmatrix}
\begin{bmatrix}
\cos \omega & -\sin \omega & 0 \\
\sin \omega & \cos \omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i & -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i & \sin \Omega \sin i \\
\sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i & -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i & -\cos \Omega \sin i \\
\sin \omega \sin i & \cos \omega \sin i & \cos i
\end{bmatrix}
\]
The R-T-N to ECI Transformation

An additional rotation gives us the R-T-N frame.

\[
R_{RTN \rightarrow ECI} = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos(\omega + f) & -\sin(\omega + f) & 0 \\ \sin(\omega + f) & \cos(\omega + f) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Where for brevity, we define \( \theta = \omega + f \). This gives us the expression

\[
\hat{e}_z = \sin i \sin(\omega + f) \hat{e}_R + \sin i \cos(\omega + f) \hat{e}_T + \cos i \hat{e}_N
\]
The R-T-N to ECI Transformation

An additional rotation gives us the R-T-N frame.

\[
\begin{bmatrix}
\cos \Omega & -\sin \Omega & 0 \\
\sin \Omega & \cos \Omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \psi & -\sin \psi \\
0 & \sin \psi & \cos \psi
\end{bmatrix}
\begin{bmatrix}
\cos (\omega + f) & -\sin (\omega + f) & 0 \\
\sin (\omega + f) & \cos (\omega + f) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Where for brevity, we define \( \theta = \omega + f \). This gives us the expression

\[
\hat{e}_z = \sin i \sin (\omega + f) \hat{e}_R + \sin i \cos (\omega + f) \hat{e}_T + \cos i \hat{e}_N
\]

Since the final rotation is just \( R_3(f) \), we combine it with the \( R_3(\omega) \) rotation so that

\[
R_3(f)R_3(\omega) = R_3(f + \omega) = R_3(\theta)
\]

similar to PC, page 200
Forces in the R-T-N Frame

\[ \vec{F} = -\mu J_2 R_e^2 \left[ \frac{3z}{r^5} \hat{e}_z + \left( \frac{3}{2r^4} - \frac{15z^2}{2r^6} \right) \hat{e}_R \right] \]

From the rotation matrices, we have that

\[ \hat{e}_z = \sin i \sin(\omega + f)\hat{e}_R + \sin i \cos(\omega + f)\hat{e}_T + \cos i\hat{e}_N \]

and since

\[ z = r \sin \phi_{gc} = r \sin i \sin(\omega + f), \]

this yields the disturbing force in the R-T-N frame:

\[ \vec{F} = \frac{-3\mu J_2 R_e^2}{r^4} \left[ \left( \frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right) \hat{e}_R + \sin^2 i \sin \theta \cos \theta \hat{e}_T + \sin i \sin \theta \cos \hat{e}_N \right] \]

\[ = \frac{-3\mu J_2 R_e^2}{r^4} \left[ \frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right]_{RTN} \]

where again, for brevity, we use \( \theta = \omega + f \)
The J2 Perturbation

The primary effect of $J_2$ is on $\Omega$ and $\omega$.

$$N = \frac{-3\mu J_2 R_e^2}{r^4} \sin i \sin(\omega + f) \cos i$$

We plug the force equations into the expressions for $\dot{\Omega}$ and $\dot{\omega}$

$$\dot{\Omega} = \sqrt{\frac{a(1 - e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i(1 + e \cos f)}$$

to get

$$\dot{\Omega} = -\frac{3\mu J_2 R_e^2}{h p^3} \cos i \sin^2(\omega + f) [1 + e \cos f]^3$$

This is the instantaneous rate of change.

- The angles $\theta$ and $f$ will cycle from $0^\circ$ to $360^\circ$ over each orbit.
- We would like to know how much of that perturbation is secular?
- What is the average over $\theta$?

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{\dot{\theta}} = \frac{\dot{\Omega}}{h/r^2}$$
The primary effect of \( J_2 \) is on \( \Omega \) and \( \omega \).

We plug the force equations into the expressions for \( \dot{\Omega} \) and \( \dot{\omega} \):

\[
\dot{\Omega} = -\frac{3\mu J_2 R^2}{e} \sin(\omega + f) \cos i 
\]

This is the instantaneous rate of change.

- The angles \( \theta \) and \( f \) will cycle from 0\(^\circ\) to 360\(^\circ\) over each orbit.
- We would like to know how much of that perturbation is secular?
- What is the average over \( \theta \)?

Recall \( \dot{\theta} = h/r^2 \) comes from equal area - equal time. \( \dot{A} = \frac{1}{2} \dot{\theta} r^2 = h/2 \).

We use the polar equation \( r = \frac{p}{1 + e \cos f} \) to eliminate \( r \).
Averaging the J2 Perturbation

Starting with

\[
\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{h/r^2} = -3J_2 \left( \frac{R_e}{p} \right)^2 \cos i \sin^2 \theta [1 + e \cos(\theta - \omega)]
\]

Then the average change over an orbit is

\[
\left. \frac{d\Omega}{d\theta} \right|_{AV} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Omega}{d\theta} d\theta = -\frac{3J_2}{2\pi} \left( \frac{R_e}{p} \right)^2 \cos i \int_0^{2\pi} \sin^2 \theta [1 + e \cos(\theta - \omega)] d\theta
\]

Now we use \( \cos(\theta - \omega) = \cos \omega \cos \theta + \sin \omega \sin \theta \) to get

\[
\int_0^{2\pi} \sin^2 \theta [1 + e \cos(\theta - \omega)] d\theta = \int_0^{2\pi} \sin^2 \theta d\theta + e \int_0^{2\pi} \sin^2 \theta \cos(\theta - \omega) d\theta
\]

\[
= \pi + e \cos \omega \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta + e \sin \omega \int_0^{2\pi} \sin^3 \theta d\theta
\]

\[
= \pi + 0 + 0 = \pi
\]

Thus, we have

\[
\left. \frac{d\Omega}{d\theta} \right|_{AV} = -\frac{3}{2} J_2 \left( \frac{R_e}{p} \right)^2 \cos i
\]
Averaging the J2 Perturbation

Given

\[ \frac{d\Omega}{d\theta} \bigg|_{AV} = -\frac{3}{2} J_2 \left( \frac{R_e}{p} \right)^2 \cos i \]

we can use the fact that

\[ n = \frac{d\theta}{dt} \bigg|_{AV} \]

to get the final expression

\[ \dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left( \frac{R_e}{p} \right)^2 \cos i \]
J2 Nodal Regression

Physical Explanation

The ascending node migrates opposite the direction of flight

\[ \dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left( \frac{R_e}{p} \right)^2 \cos i \]

The equatorial bulge produces extra pull in the equatorial plane

- Creates an averaged torque on the angular momentum vector
- Like gravity, the torque causes \( \vec{h} \) to precess.
- Only depends on inclination
  - Also \( a \) and \( e \)...

Image credit: Vallado
The nodal regression rate is often large. **Cannot Be Neglected!!!.**

**Figure:** Magnitude of Regression Rate vs. inclination and altitude.

**Fig. 10.2** Regression rate due to oblateness vs inclination for various values of average altitude.

**Figure:** Magnitude of Regression Rate vs. inclination and altitude
Repeating Ground Tracks

\( \dot{\Omega} \) has a large effect on the design of *Repeating Ground Tracks*.

- The rotation of the earth over an orbit is given by
  \[
  \Delta L_1 = -2\pi \frac{T}{24 \times 3600} = -2\pi \frac{2\pi \sqrt{a^3}}{24 \times 3600}
  \]

- The change in \( \Omega \) over an orbit is
  \[
  \Delta L_2 = -\frac{3\pi J_2 R_e^2 \cos(i)}{a^2 (1 - e^2)^2}
  \]

- For a ground track to repeat, we require
  \[
  j \mid \Delta L_1 + \Delta L_2 \mid = j \left| -2\pi \frac{2\pi \sqrt{a^3}}{T E} - \frac{3\pi J_2 R_e^2 \cos(i)}{a^2 (1 - e^2)^2} \right| = k2\pi
  \]
  for some integers \( j \) and \( k \).

- \( j \) is the number of orbits before repeat.
- \( k \) is the number of days (sidereal) before repeat.
Repeating Ground Tracks

- The rotation of the earth over an orbit is given by
  \[ \Delta L_1 = -2\pi \frac{T}{24 \times 3600} = -2\pi \frac{T}{24 \times 3600} \]

- The change in \( \Omega \) over an orbit is
  \[ \Delta \Omega = -3\pi J_2 R e \cos(i) a^2 (1 - e^2) \]

For a ground track to repeat, we require

\[ j |\Delta L_1 + \Delta \Omega| = j \left| -2\pi \frac{T}{24 \times 3600} - 3\pi J_2 R e \cos(i) a^2 (1 - e^2) \right| = 2\pi \]

for some integers \( j \) and \( k \).

- \( j \) is the number of orbits before repeat.
- \( k \) is the number of days (sidereal) before repeat.

Figure: SZ-4 Repeating ground track (Sven’s Space Place)
J2 Apsidal Rotation

Recall the Argument of Perigee Equation:

\[
\dot{\omega} = -\dot{\Omega} \cos i + \sqrt{a(1 - e^2)} \frac{\sqrt{e^2 \mu}}{e^2} \left( -R \cos f + T \frac{(2 + e \cos f) \sin f}{1 + e \cos f} \right)
\]

\[
R = -\frac{3 \mu J_2 R_e^2}{r^4} \left( \frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right), \quad T = -\frac{3 \mu J_2 R_e^2}{r^4} \sin^2 i \sin \theta \cos \theta
\]

The argument of perigee (\(\omega\)) is linked to RAAN (\(\Omega\)). The average value is

\[
\frac{d\omega}{d\theta} = -\frac{d\Omega}{d\theta} \cos i + \frac{3 J_2 R_e^2}{2p^2} \left[ 1 - \frac{3}{2} \sin^2 i \right]
\]

where

\[
\frac{d\Omega}{d\theta} \cos i = -\frac{3}{2} J_2 \left( \frac{R_e}{p} \right)^2 \cos^2 i
\]

\[
= -\frac{3}{2} J_2 \left( \frac{R_e}{p} \right)^2 (1 - \sin^2 i)
\]

Image credit: Vallado
J2 Apsidal Rotation

Recall the Argument of Perigee Equation:

\[ \dot{\omega} = -\dot{\Omega} \cos i + \sqrt{a (1 - e^2)} e^2 \mu \left( -3 \cos \cos f + \frac{1}{1 + \cos f} \right) \sin f \]

\[ R = -\frac{3 \mu J^2}{r^3} \left( \frac{1}{2} - \frac{3}{2} \sin^2 i \right) \]

\[ T = -\frac{3 \mu J^2}{r^3} \sin^2 \sin \theta \cos \theta \]

The argument of perigee (\( \omega \)) is linked to RAAN (\( \Omega \)). The average value is

\[ \frac{d\omega}{d\theta} \cos i = -\frac{3}{2} J_i \left( \frac{R}{r} \right)^2 \cos^3 i \]

\[ = -\frac{3}{2} J_i \left( \frac{R}{r} \right)^2 (1 - \sin^2 i) \]

There are 3 parts acting here

- If the perigee were fixed in space, \( \dot{\Omega} \) would shorten the angle to this point.
- A tangential component advances perigee
- A radial component pulls perigee forward in the orbit.
J2 Apsidal Rotation

Similar to nodal regression, but perigee moves forward or backward, depending on inclination.

\[ \dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left( \frac{R_e}{p} \right)^2 \left[ 2 - \frac{5}{2} \sin^2 i \right] \]
The apsidal rotation rate is often large.

\[
\dot{\omega} = \frac{9.9639}{(1 - e^2)^2} \left( \frac{R}{R + h} \right)^{7/2} (2 - \frac{5}{2} \sin^2 i) \text{ deg/Mean Solar Day}
\]

\[
\bar{h} = \frac{h_{\text{APOGEE}} + h_{\text{PERIGEE}}}{2}
\]

\[
e = \frac{h_a - h_p}{h_a + h_p + 2R}
\]

\[
R = 3444 \text{ nmi}
\]

\[
63.4 \text{ deg or 116.5 deg}
\]

Figure: Magnitude of Regression Rate vs. inclination and altitude
The $J_2$ effect on other elements is usually minor. $\dot{a} \approx 0$.

**Figure:** Eccentricity Change for Low-Inclination Orbit
J2 Effect
Other Elements: Eccentricity

Figure: Eccentricity Change for Moderate-Inclination Orbit
"Frozen Orbits" can be designed to minimize changes in eccentricity

- Use the $J_3$ perturbation (Not covered here)
- Require particular choices of $e$ and $\omega$
J2 Effect

Other Elements: Inclination

Fig. 10.6  Inclination variation without correction (5:30 orbit).

Figure: Inclination Change for Eccentric and Circular Orbits
\[ \dot{\omega}_{J_2,av} = \frac{3}{2} n J_2 \left( \frac{R_e}{p} \right)^2 \left[ 2 - \frac{5}{2} \sin^2 i \right] \]

**Definition 3.**

A **Critically Inclined Orbit** is one where \( \dot{\omega} = 0 \)

For a critically inclined orbit,

\[ 4 - 5 \sin^2 i = 0 \]

which means

\[ i = \sin^{-1} \sqrt{\frac{4}{5}} \]

\[ = 63.43^\circ \quad \text{or} \quad 116.57^\circ \]
Definition 3. A Critically Inclined Orbit is one where \( \dot{\omega} = 0 \).
For a critically inclined orbit,
\[
4 - 5 \sin^2 i = 0
\]
which means
\[
i = \sin^{-1} \sqrt{\frac{4}{5}} = 63.43^\circ \quad \text{or} \quad 116.57^\circ.
\]

\( J_2 \) Special Orbits

**Figure:** Molniya Orbit

**Figure:** Tundra Orbit
Sun-Synchronous orbits maintain the same orientation of the orbital plane with respect to the sun.

Applications:
- Mapping
- Solar-Powered
- Shadow-evading
- Time-of-Day Apps
The earth rotates $360^\circ$ about the sun every 365.25 days.

**Definition 4.**

A **Sun-Synchronous Orbit** is one where $\dot{\Omega} = .9855^\circ/day = 1.992 \cdot 10^{-7} \text{rad/s}$.

Thus

$$\cos i = -1.992 \cdot 10^{-7} \left(\frac{\rho}{R_e}\right)^2 \frac{2}{3nJ_2}$$

- The orbital plane rotates once every year.
$J_2$ Special Orbits
Sun-Synchronous Orbits

Unlike critically inclined orbits, sun-synchronous orbits depend on altitude.
**Problem:** Design a sun-synchronous orbit with $r_p = R_e + 695\text{km}$ and $r_a = R_e + 705\text{km}$.

**Solution:** The desired inclination for a sun-synchronous orbit is given by

$$i = \cos^{-1} \left( 5.02 \cdot 10^6 \left( \frac{p}{R_e} \right)^2 \frac{2}{3nJ_2} \right)$$

For this orbit $a = R_e + 700\text{km} = 7078\text{km}$. The eccentricity is

$$e = 1 - \frac{r_p}{a} = \cdot00071$$

Thus $p = a(1 - e^2) = 6999.65\text{km}$. $n = \sqrt{\frac{\mu}{a^3}} = .0011$. Finally, $J_2 = .0010826$. Thus the required inclination is

$$i = 1.716\text{rad} = 98.33^\circ$$
Problem: Molniya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 24 hours (actually Tundra orbit) and which precesses at $\dot{\Omega} = -0.2^\circ$/day.

Solution: We can first use the period to solve for $a$. From

$$n = \sqrt{\frac{\mu}{a^3}} = 7.27 \cdot 10^{-5}$$

and $n = 2\pi/T = 2\text{rad/day}$ we have

$$a = \sqrt[3]{\frac{\mu}{n^2}} = 42,241\text{km}$$

Now the critical inclination for $\dot{\omega} = 0$ is $i = 63.4^\circ$ or $i = 116.6^\circ$. Since $\dot{\Omega} < 0$, we must choose $i = 63.4^\circ$. To achieve $\dot{\Omega} = -0.2^\circ$/day, we use

$$\dot{\Omega} = -\frac{3nJ_2R_e^2}{2a^2(1-e^2)^2} \cos i$$
Numerical Example

Molniya Orbit

Problem: Molniya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 24 hours (actually Tundra orbit) and which precesses at $\dot{\Omega} = -2^{\circ}/\text{day}$.

Solution: We can first use the period to solve for $a$. From

$$n = \sqrt{\frac{\mu}{a^3}} = \frac{7.27 \times 10^{-5}}{2\pi/T} = \frac{2\pi}{2\pi} \text{rad/day} = 1.0 \text{rad/day}$$

we have

$$a = \sqrt{\frac{\mu}{n^2}} = 42,241 \text{km}$$

Now the critical inclination for $\dot{\omega} = 0$ is $i = 63.4^{\circ}$ or $i = 116.6^{\circ}$. Since $\dot{\Omega} < 0$, we must choose $i = 63.4^{\circ}$. To achieve $\dot{\Omega} = -2^{\circ}/\text{day}$, we use

$$\dot{\Omega} = -\frac{3nJ_2}{2a^2(1-e^2)^{3/2}} \sin i$$

Northern Molniya orbits have an argument of perigee of $+90^{\circ}$.

- Used for sensing and communication.

- Geosynchronous orbits cannot communicate well with or observe locations at high latitude.

- Molniya orbits launched from high latitude do not require large inclination changes after launch, unlike geosynchronous orbits.

- Provides continuous coverage with 3 satellites.

- Also used for US-observing spy sats and early-warning sats.

- Example of a semi-synchronous frozen tundra orbit with repeating ground track.
Numerical Example

Molnaya Orbit, continued

Since \( a \) is already fixed, we must use \( e \). We can solve for \( e \) as

\[
e = \sqrt{1 - \sqrt{-\frac{3nJ_2R_e^2}{2\dot{\Omega}a^2}\cos i}} = .7459.
\]

Note: Make sure the units of \( a \) and \( n \) match those of \( R_e \) and \( \dot{\Omega} \), respectively.
Summary

This Lecture you have learned:

How to account for perturbations to Earth gravity

- Gravity Mapping
- Harmonic Functions
- $J_2$ Perturbation
  ▶ Effect on $\Omega$
  ▶ Effect on $\omega$
  ▶ Minor effect ($e, i$)

How to design specialized orbits

- Critically - Inclined Orbit.
- Sun-Synchronous Orbit.
- Applications

Next Lecture: Interplanetary Mission Planning.