Spacecraft Dynamics and Control

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Lecture 14: Interplanetary Mission Planning

Introduction

In this Lecture, you will learn:

Sphere of Influence

- Definition

Escape and Re-insertion

- The light and dark of the Oberth Effect

Patched Conics

- Heliocentric Hohmann

Planetary Flyby

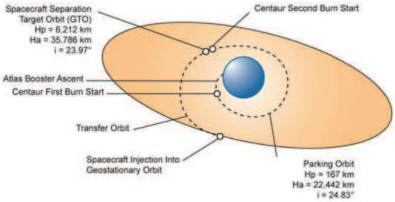
- The Gravity Assist

The Sphere of Influence Model

Simplifying Three-Body Motion

Consider a Simple Earth-Moon Trajectory.

1. Launch
2. Establish Parking Orbit
3. Escape Trajectory
4. Arrive at Destination
5. Circularize or Depart Destination



The big difference is that now there are 3 bodies.

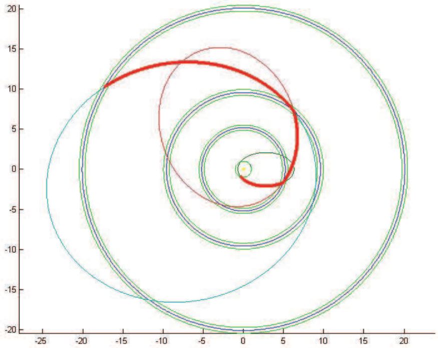
- We only know how to solve the 2-body problem.
- Solving the 3-body problem is beyond us.

Patched Conics

For interplanetary travel, the problem is even more complicated.

Consider the Figure

- The motion is elliptic about the sun.
- The motion is affected by the planets
  - Interference only occurs in the green bands.
  - Motion about planets is hyperbolic.
  - Direction and Magnitude of $\vec{v}$ changes.



The solution is to break the mission into segments.

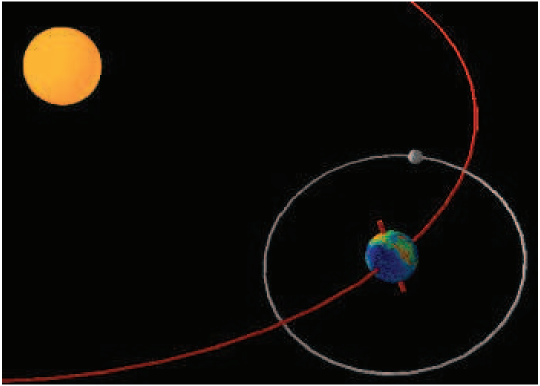
- During each segment we use *two-body motion*.
- The third body is a disturbance.

Question: Who is in charge??

- The Sphere of Influence of A stops when A is no longer the dominant force.
- What do we mean by dominant?

Wrong Definition:
The Sphere of Influence of A is the region wherein A exerts the largest gravitational force.

Why Wrong?
This would imply the moon is not in earth's Sphere of Influence!!!



Sun Perspective: Lets group the forces as central and disturbing.
Consider motion of a spacecraft relative to the sun:

$$\ddot{\vec{r}}_{sv} + \underbrace{Gm_s \frac{\vec{r}_{sv}}{\|\vec{r}_{sv}\|^3}}_{\text{Effect of sun on object}} = -Gm_p \left[\underbrace{\frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3}}_{\text{Effect of planet on object}} + \underbrace{\frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3}}_{\text{Effect of planet on sun}} \right]$$

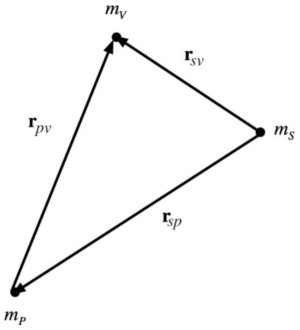
where p denotes planet, v denotes vehicles and s denotes sun.

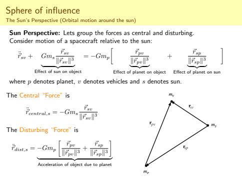
The Central "Force" is

$$\ddot{\vec{r}}_{central,s} = -Gm_s \frac{\vec{r}_{sv}}{\|\vec{r}_{sv}\|^3}$$

The Disturbing "Force" is

$$\ddot{\vec{r}}_{dist,s} = \underbrace{-Gm_p \left[\frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3} + \frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3} \right]}_{\text{Acceleration of object due to planet}}$$





- For the sun-moon system, e.g., the vectors

$$\frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3} >> \frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3} \cong 0$$

  so

$$\frac{\ddot{\vec{r}}_{dist,s}}{\ddot{\vec{r}}_{central,s}} \cong \frac{m_p}{m_s} \frac{\|\vec{r}_{sv}\|^2}{\|\vec{r}_{pv}\|^2}$$

- So if $\|\vec{r}_{pv}\|$ is small and $\|\vec{r}_{sv}\|$ is big, the disturbing force dominates.

Planet Perspective: The relative motion of the spacecraft with respect to the planet is

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The Central "Force" for the planet is

$$\ddot{\vec{r}}_{central,p} = -Gm_p \frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3}$$

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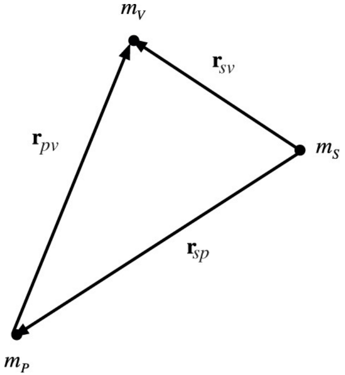

Sphere of influence

The Planet's Perspective (Orbit around the planet)

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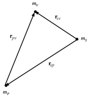

- When the vehicle is near the planet, $\vec{r}_{sp} \cong \vec{r}_{sv}$ and hence

  $$\frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3} \cong \frac{\vec{r}_{sv}}{\|\vec{r}_{sv}\|^3}$$

  so $\ddot{\vec{r}}_{dist,p} \cong 0$ and

  $$\frac{\ddot{\vec{r}}_{dist,p}}{\ddot{\vec{r}}_{central,p}} \cong \frac{m_s}{m_p} \cdot 0 \cong 0$$

  and hence the relative size of the disturbance is small.
- Sphere of influence is based on the relative distance.

Definition 1.

An object is in the **Sphere of Influence**(SOI) of body 1 if

$$\frac{\|\ddot{\vec{r}}_{dist,1}\|}{\|\ddot{\vec{r}}_{central,1}\|} < \frac{\|\ddot{\vec{r}}_{dist,2}\|}{\|\ddot{\vec{r}}_{central,2}\|}$$

for any other body 2.

That is, the ratio of disturbing “force” to central “force” determines which planet is in control.

For planets, an approximation for determining the SOI of a planet of mass m_p at distance d_p from the sun is

$$R_{SOI} \cong \left(\frac{m_p}{m_s}\right)^{2/5} d_p$$

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$$\frac{m_p}{m_s}\frac{\|\vec{r}_{sv}\|^2}{\|\vec{r}_{pv}\|^2} > \frac{m_s\left[\frac{\vec{r}_{sv}}{\|\vec{r}_{sv}\|^3} - \frac{\vec{r}_{sp}}{\|\vec{r}_{sp}\|^3}\right]}{m_p\frac{\vec{r}_{pv}}{\|\vec{r}_{pv}\|^3}} \cong \frac{m_s\left[\vec{r}_{sv} - \vec{r}_{sp}\right]}{m_p\frac{\vec{r}_{pv}\|\vec{r}_{sv}\|^3}{\|\vec{r}_{pv}\|^3}}$$

$$\frac{m_p^2}{m_s^2}\frac{\|\vec{r}_{sv}\|^5}{\|\vec{r}_{pv}\|^5} > \frac{\left[\vec{r}_{sv} - \vec{r}_{sp}\right]}{\vec{r}_{pv}} \cong 1$$

$$\frac{m_p^2}{m_s^2}\|\vec{r}_{sv}\|^5 > \|\vec{r}_{pv}\|^5$$

$$\|\vec{r}_{pv}\| < \left(\frac{m_p}{m_s}\right)^{2/5}\|\vec{r}_{sv}\|$$

Sphere of influence

Table 7.1 Sphere of Influence Radii

Celestial Body	Equatorial Radius (km)	SOI Radius (km)	SOI Radius ($body\ radii$)
Mercury	2487	1.13×10^5	45
Venus	6187	6.17×10^5	100
Earth	6378	9.24×10^5	145
Mars	3380	5.74×10^5	170
Jupiter	71370	4.83×10^7	677
Neptune	22320	8.67×10^7	3886
Moon	1738	6.61×10^4	38

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- The sphere of influence of a planet is defined w/r another mass.
- Distance from earth to the moon is 385,000km
- e.g. Note that sphere of influence of the Moon (w/r to the earth) is inside the sphere of influence of the Earth (w/r to the sun)!
- The SOI of the earth w/r to the moon is different that the SOI w/r to the sun!

Pluto's sphere of influence is generally considered to be 4.2 million km or 3,650 body radii.

Example: Lunar Lander

Problem: Suppose we want to plan a lunar-lander mission. Determine the spheres of influence to consider for a patched-conic approach.

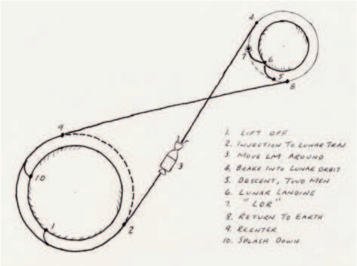
- The SOI of the earth is of radius 924,000km.
- The SOI of the moon is of radius 66,100km.

Solution: The moon orbits at a distance of 385,000km. The spacecraft will transition to the lunar sphere at distance

$$r = 385,000 - 66,100 = 318,900km$$

We will probably also need a plane change. A reasonable mission design is

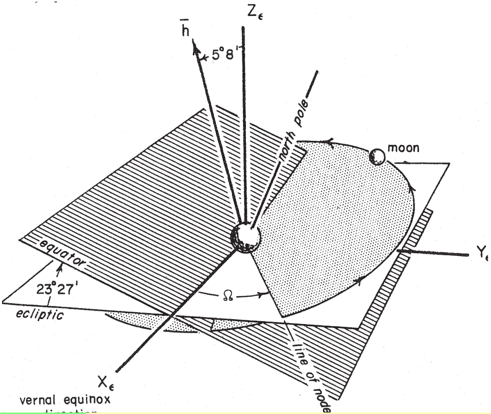
1. Depart earth on a Hohmann transfer to radius $317,900$ km.
2. Perform inclination change near apogee.
3. Enter sphere of influence of the moon.
4. Establish parking orbit.

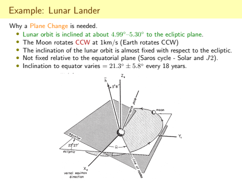


Example: Lunar Lander

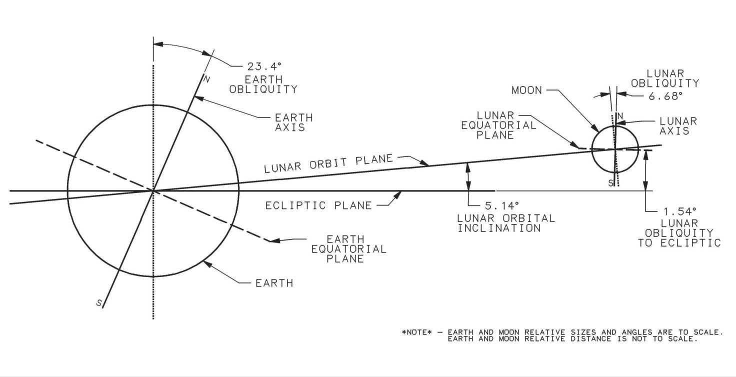
Why a Plane Change is needed.

- Lunar orbit is inclined at about $4.99°–5.30°$ to the ecliptic plane.
- The Moon rotates CCW at 1km/s (Earth rotates CCW)
- The inclination of the lunar orbit is almost fixed with respect to the ecliptic.
- Not fixed relative to the equatorial plane (Saros cycle - Solar and $J2$).
- Inclination to equator varies $= 21.3° \pm 5.8°$ every 18 years.



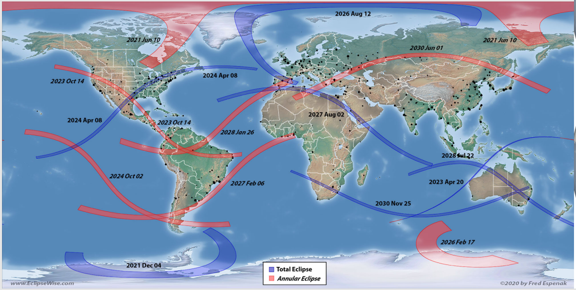


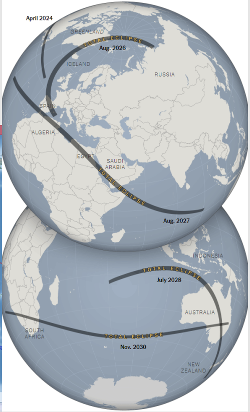
- The orbit of the moon is significantly perturbed by the sun.
- Somewhat similar to J2 perturbation, but centered on ecliptic.
- RAAN of lunar orbit processes with period of 18 years.



More Illustrations of the Lunar Orbit

Motion during Eclipse:





5 Stage Lunar Intercept Mission

First Stage Lunar Tug Assist

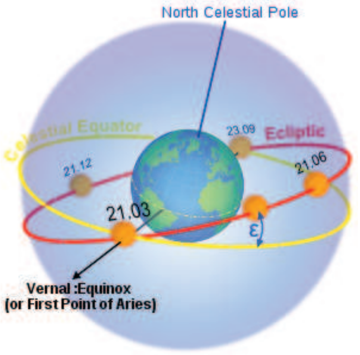
Stages of Interplanetary Mission Planning

1. Establish Orbit in Ecliptic Plane (Low Earth Orbit) with counter-clockwise rotation
2. Burn to escape with excess velocity v_∞
3. Establishes Velocity in Solar Frame
   3.1 $v_p = v_e + v_\infty$ for dark-side burn (Outer planets)
   3.2 $v_a = v_e - v_\infty$ for light-side burn (Inner planets)
4. Propagate Hohman (or Lambert) to destination
   4.1 Find v_a for outer planets
   4.2 Find v_p for inner planets
5. Compute relative velocity (v_r) in planet (Venus) frame $v_r = \|v_p - v_v\|$
   5.1 For flyby, use targeting radius to find turning angle.
   5.2 For insertion, use targeting radius to find r_p.
6. Compute post-flyby relative velocity and convert to Heliocentric frame.

Problem: Design an Earth-Venus rendez-vous. Final orbit around Venus should be posigrade and have altitude 500km.

First Step: Align parking orbit with ecliptic plane.

- All planets move in the ecliptic plane
  - $i \cong 23.4^{\circ}$
- Circular orbit.
  - Radius $r \cong 6578km$



Stages of Interplanetary Mission:

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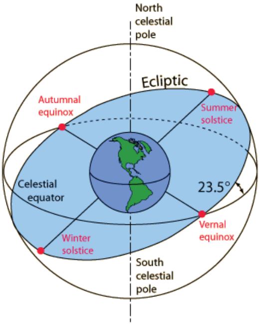
Moving to the Ecliptic Plane

All planets in the solar system orbit the sun in the ecliptic plane.

- Transition must occur when the orbital plane and ecliptic planes intersect.



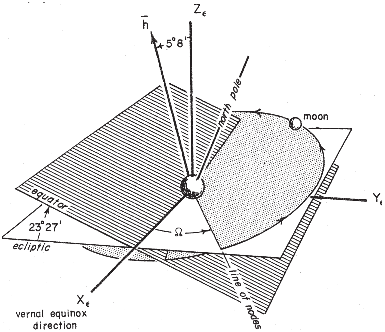
Any earth-centered orbit passes through the ecliptic twice per orbit.

- But not at the ascending node (w/r to the equatorial plane).
- But not at the correct time (f??).

Transition to the ecliptic

To change to the ecliptic plane:

- Burn at ascending node w/r to the ecliptic plane.
- Execute a plane change.

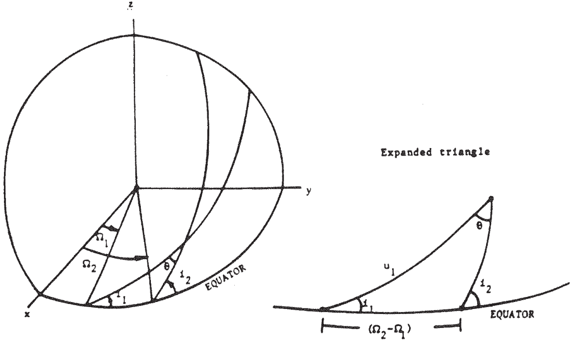


Requires a change in both Ω and i

- New $\Omega = 0$
- New $i = 23.27°$

Interplanetary Hohmann Transfer

Transition to the ecliptic



Our desired orbit has

- $i_2 = \epsilon = 23.5^\circ$ - Inclination to the ecliptic
- $\Omega_2 = 0^\circ$ - by definition: Ω is measured from FPOA (intersection of equatorial and ecliptic planes).

If our initial orbit has inclination i_1 and RAAN Ω_1, then the angle change is

$$\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1)$$

Interplanetary Hohmann Transfer

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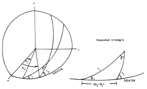

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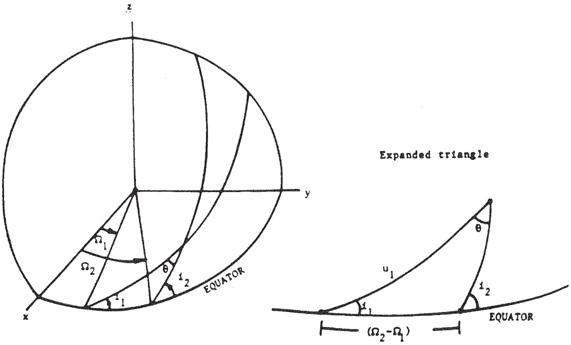
- It is not possible to launch directly into the ecliptic from the U.S. (Recall for Kennedy $\phi_{gc} = 28.5°$)
- However, we may choose launch time θ_{LST} in order to select RAAN Ω_1
- For the ecliptic, $i_2 = 23.5°$.
- For Kennedy, $i_1 = 28.5°$
- For the ecliptic plane, $\Omega_2 = 0°$.
- To minimize Δv, we want to minimize θ. To do this, we may select $\Omega_1 = 0°$, which yields

$$\theta = \cos^{-1}\left(\cos(28.5°)\cos(23.5°) + \sin(28.5°)\sin(23.5°) * \cos(0°)\right) = 5°$$

- If combined with a burn to escape, the Δv for a $5°$ plane change is almost negligible!



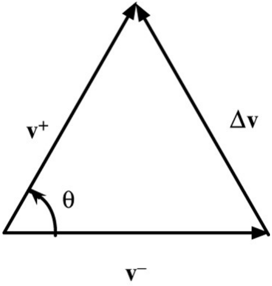
The position in the orbit is given by

$$\cos(\omega + f) = \frac{\cos i_1 \cos\theta - \cos i_2}{\sin i_1 \sin\theta}$$

Where recall

- $i_2 = \epsilon = 23.5^\circ$

The Plane Change



The Δv required required for the plane change is then

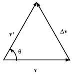
$$\Delta v = 2v \sin\frac{\theta}{2}$$

or

$$\Delta v^2 = v(t_k^-)^2 + v(t_k^+)^2 - 2v(t_k^-)v(t_k^+)\cos\Delta\theta$$

if combined with a velocity change ($v(t_k^-)$ to $v(t_k^+)$).

The Plane Change



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In truth, we try and avoid large plane changes. Typically, it is better to launch directly into the ecliptic plane. This is normally possible if the launch site is below 23.5° latitude and the launch time is carefully chosen.

Stage 2: Escape Trajectory

Step 2a: Design an Interplanetary Hohmann Transfer

We need the magnitude and direction of velocity in the Heliocentric Frame.

The perigee and apogee velocities of the Heliocentric transfer ellipse are

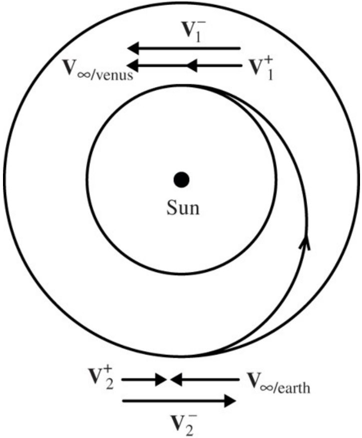
$$v_1^+ = \boxed{v_p = \sqrt{2\mu_{sun}\frac{r_e}{r_v(r_e+r_v)}} = 37.73 km/s}$$

$$v_2^+ = \boxed{v_a = \sqrt{2\mu_{sun}\frac{r_v}{r_e(r_e+r_v)}} = 27.29 km/s}$$

Where

- r_e is dist. from sun to earth ($v_e = 29.8$)
- r_v is dist. from sun to venus ($v_v = 35.1$)

Because Venus is an inner planet, apogee velocity occurs at Earth



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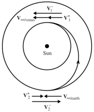

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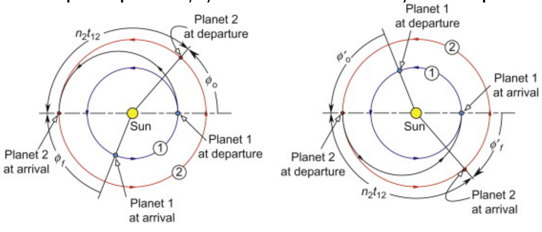
Phasing of the Hohmann Transfer

If you want to intercept a planet, you have to time your departure!



(a) (b)

The transfer orbit sweeps $180°$ in time $\Delta T = \pi\sqrt{\frac{a^3}{\mu}} = \pi\sqrt{\frac{(r_1+r_2)^3}{\mu}}$. During this time, the planet will sweep an angle of

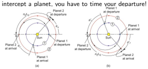
$$D\theta = 360° \frac{\Delta T}{T_{planet}} = 360° \frac{\pi\sqrt{\frac{(r_1+r_2)^3}{\mu}}}{\pi\sqrt{\frac{r_2^3}{\mu}}} = 360° \sqrt{\frac{(r_1+r_2)^3}{r_2^3}}$$

So you want your relative angle at departure to be

$$\phi_0 = 180° - 360° \sqrt{\frac{(r_1+r_2)^3}{r_2^3}}$$

Phasing of the Hohmann Transfer

If you want to intercept a planet, you have to time your departure!



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So you want your relative angle at departure to be

$$\phi_0 = 180° - 360° \sqrt{\frac{(r_1+r_2)^3}{r_2^3}}$$

The relative angle between two planets changes at rate

$$n_{rel} = n_1 - n_2$$

So, if you miss your launch, you will have to wait for

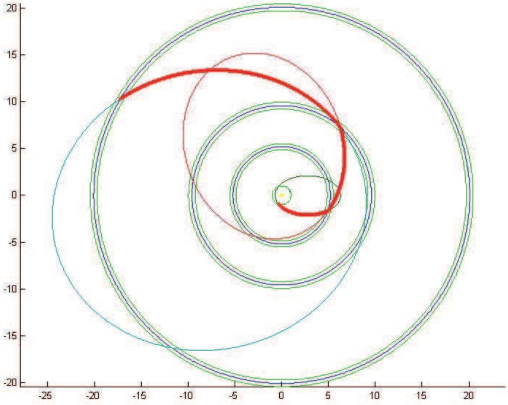
$$T_{syn} = \frac{2\pi}{n_{rel}}$$

This is known as the **synodic period**. The synodic period for: Mercury: 88 days; Venus: 225 days; Mars: 2.1 years; Jupiter: 11.9 years; Saturn: 29.7 years; Uranus: 84.0 years; Neptune: 164.8 years.

Step 2: Interplanetary Hohmann Transfer

We can use the Hohmann transfer (2-body, Elliptic orbits) because the voyage will take place almost exclusively in the sun's sphere of influence.

- The earth orbits at radius $1au = 1.5 \cdot 10^8 km = 23,518 ER$.
- The SOI of the earth is only $145 ER$, or $.5\%$.



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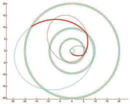


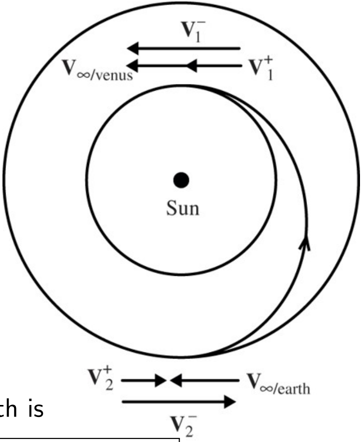
- None of the trajectories in this diagram are Hohmann transfers (although the first is nearly so)
- The phasing must be perfect for a Hohman transfer, and so these are only possible for single-planet routes, with no gravity assist.
- The Δv at planet 2 to intersect planet 3 is chosen by solving **Lambert's Problem**.

Problem: We need to know the Δv magnitude relative to earth's motion.

- $v_a = v_2^+$ is w/r to inertial frame.
- Earth is moving in the inertial frame.
  - The earth frame is moving with velocity

$$v_2^- = v_e = \sqrt{\frac{\mu_s}{\|\vec{r}_{se}\|}} = 29.78 km/s$$

- What is this v_a velocity relative to earth?



We have

$$v_2^+ = v_a = v_2^- + v_{\infty,e}$$

Thus our desired velocity with respect to the earth is

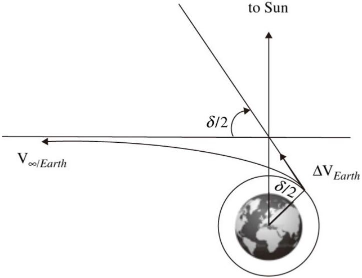
$$\Delta v_e = \boxed{v_{\infty,e} = v_a - v_e^- = 27.29 - 29.78 = -2.49 km/s}$$

- The magnitude of Δv_e is determined by *excess velocity*
- The direction of Δv_e is determined by timing

Problem: How to achieve the initial $v_{\infty,e} = -2.49 km/s$?

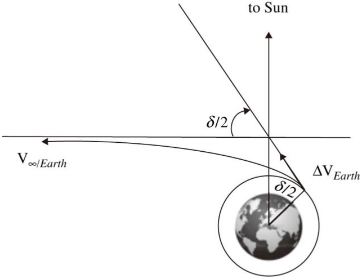
- We need to escape earth orbit.
- Must have leftover velocity (excess velocity) of $2.49 km/s$.
  - Implies the total energy (w/r to the earth) after burn is

$$E_+ = \frac{1}{2} v_{\infty,e}^2 = 3.1223$$



Interplanetary Hohmann Transfer

Suppose the spacecraft is in a circular parking orbit of radius $r_{park} = 6578$km.



- The velocity before the burn will be

$$v_{\text{park}} = \sqrt{\frac{\mu_e}{r_{\text{park}}}} = 7.7843km/s$$

- The velocity after burn (v_{after}) can be found by solving the energy equation.

$$E = \frac{1}{2}v_{\text{after}}^2 - \frac{\mu_e}{r_{\text{park}}} = E_+ = +3.1223$$

  Solving for v_{after}, we get

$$v_{\text{after}} = \sqrt{2E + 2\frac{\mu_e}{r_{\text{park}}}} = \sqrt{v_{\infty,e}^2 + 2\frac{\mu_e}{r_{\text{park}}}} = 11.288km/s$$

- This yields a Δv_{local} of

$$\boxed{\Delta v_{\text{local}} = v_{\text{after}} - v_{\text{park}} = 3.5044km/s}$$

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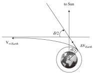

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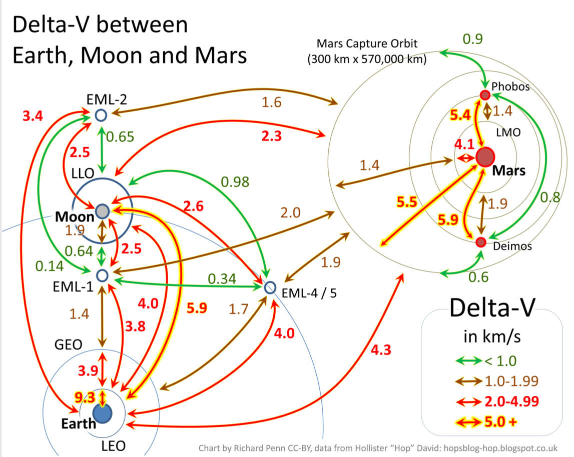
Note that $\Delta v = 3.5$ km/s is less than the Δv to reach GEO.

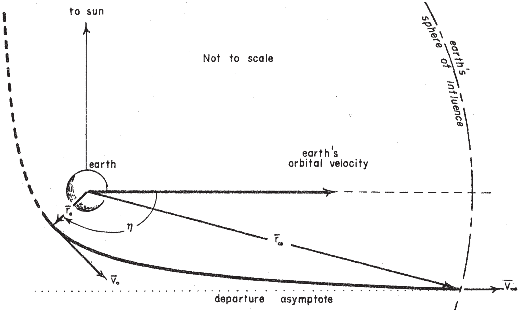


Light Side or Dark Side Departure?

Getting the Sign (direction, $\pm$) right

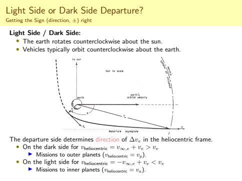
Light Side / Dark Side:

- The earth rotates counterclockwise about the sun.
- Vehicles typically orbit counterclockwise about the earth.



The departure side determines direction of Δv_e in the heliocentric frame.

- On the dark side for $v_{\text{heliocentric}} = v_{\infty,e} + v_e > v_e$
  - Missions to outer planets ($v_{\text{heliocentric}} = v_p$).
- On the light side for $v_{\text{heliocentric}} = -v_{\infty,e} + v_e < v_e$
  - Missions to inner planets ($v_{\text{heliocentric}} = v_a$).



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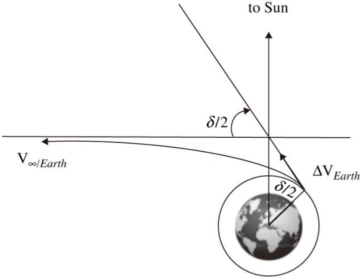
Interplanetary Hohmann Transfer

When to make the burn?

Timing: The Δv should occur at $\delta/2$ before midnight/noon, where δ is the turning angle

$$\delta = 2\sin^{-1}\frac{1}{e}$$



Eccentricity (e) can be found as:

- Energy: $E = \frac{1}{2}v_{\infty,e}^2 = 2.067 = -\frac{\mu}{2a}$ yields

$$a = -\frac{\mu}{v_{\infty,e}^2} = -\frac{\mu}{2E} = -96,420km$$

- Perigee: $r_{p,e} = r_c = a(1-e) = 6578km$ yields

$$\boxed{e = 1 - \frac{r_{p,e}}{a} = 1.0682}$$

This yields a turning angle of

$$\boxed{\delta = 2.423rad = 138.83^\circ}$$

Thus the spacecraft should depart at $\delta/2 = 69.4^\circ$ before noon.

Arrival at Venus

At arrival, our excess velocity w/r to Venus ($v_{\infty,v}$) will be

$$v_{\infty,v} = v_p - v_v = v_1^- - v_1^+ = 37.81km/s - 35.09km/s = 2.71km/s$$

where

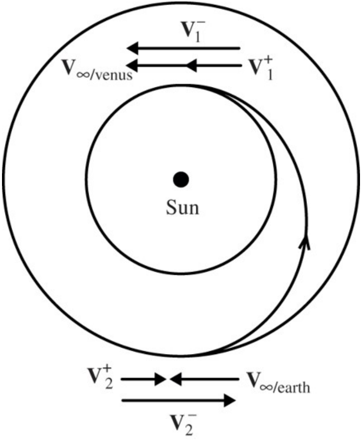
- $v_1^+ = v_v$ is the velocity of venus

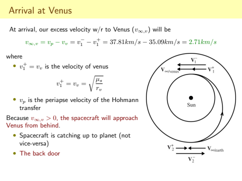
$$v_1^+ = v_v = \sqrt{\frac{\mu_s}{r_v}}$$

- v_p is the periapse velocity of the Hohmann transfer

Because $v_{\infty,v} > 0$, the spacecraft will approach Venus from behind.

- Spacecraft is catching up to planet (not vice-versa)
- The back door





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Arrival at Venus

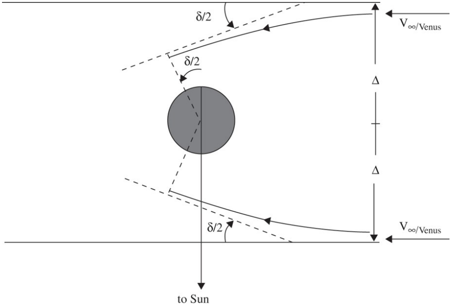
Venus Data:

$$R_v = 6187km, \quad \mu_v = 324859, \quad a_{\text{venus}} = 1.08 \cdot 10^8$$

Desired Orbit: Circular, posigrade (counterclockwise) with

$$r_c = 6187 + 500 = 6687km$$

For a **Counterclockwise** orbital insertion from the **Back Door**, we want to approach Venus on the **Dark Side**



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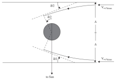


- If we were travelling to an outer planet, we are using the **Front Door** and hence would approach on the **Light Side** to achieve a **Counterclockwise** orbit
- This is because for outer planets, we are moving slower than the planet
- Hence the planet is approaching us.
- We would enter the SOI from the left.

Arrival at Venus

For orbital insertion, we want to perform a retrograde burn at periapse of the incoming hyperbola.

To achieve a circular orbit of radius $r_c = 6687km$, we need the periapse of our incoming hyperbola to occur at

$$r_{p,v} = a(1-e) = 6687km.$$

The energy of the incoming hyperbola is given by the excess velocity as

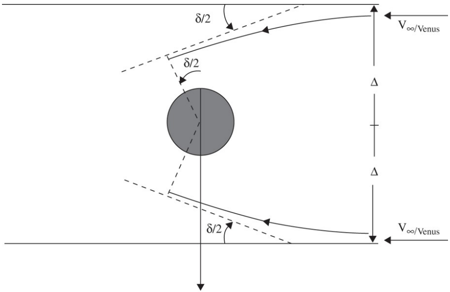
$$E = \frac{1}{2}v_{\infty,v}^2 = 3.67.$$

This fixes the semimajor axis at

$$\boxed{a = -\frac{\mu_v}{v_{inf,v}^2} = -44,232km.}$$

Thus to achieve $r_p = a(1-e)$, we need

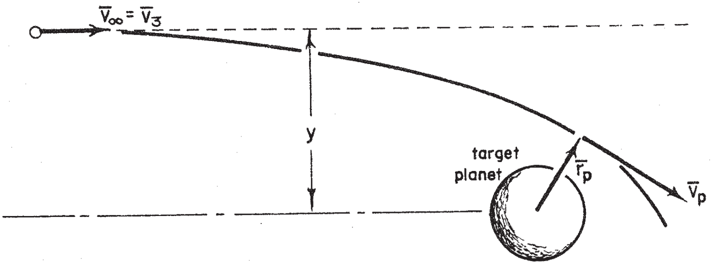
$$\boxed{e = 1 - \frac{r_p}{a} = 1.15.}$$



Arrival at Venus

To achieve the desired $e = 1.15$, we control the conditions at the *Patch Point*.

- We do this through the angular momentum, h.



We can control the **Target Radius**, Δ through small adjustments far from the planet. Angular momentum can be exactly controlled through target radius, Δ.

$$h_v = v_{\infty,v}\Delta$$

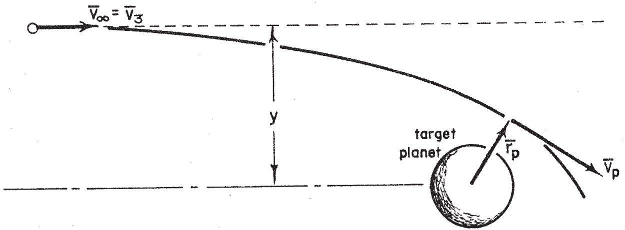
Arrival at Venus

Solution: For a given a, e is determined by $p = a(1-e^2)$.

- But p is defined by angular momentum (and thus target radius).

$$p = \frac{h^2}{\mu_v} = \frac{\Delta^2 v_{\infty,v}^2}{\mu_v}$$

- For $a = -44,232km$ and $e = 1.15$, we get $p = 14,265km$.



Given a desired p we solve for target radius, Δ,

$$\boxed{\Delta = \sqrt{\frac{p\mu_v}{v_{\infty,v}^2}} = \sqrt{\frac{a(1-e^2)\mu_v}{v_{\infty,v}^2}} = 25,120km}$$

Injection into Circular Orbit

Finally, we need to slow down to achieve circular orbit.

- The velocity at periapse (6687km) is given by the vis-viva equation.

$$v = \sqrt{\frac{2\mu_v}{r_{p,v}} - \frac{\mu_v}{a}} = 10.223km/s$$

- The velocity of a circular orbit is

$$v_c = \sqrt{\mu_v} r_{p,v} = 6.97km/s$$

Thus the Δv required to circularize the orbit is

$$\Delta v = 6.97 - 10.223 = -3.253km/s$$

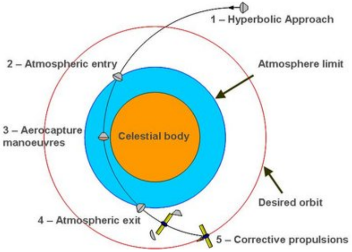


Figure: Aerobraking can also assist with Δv

Alternatively, for simple planetary capture:

Escape Velocity at 6687: $v_{esc} = \sqrt{\frac{2\mu_v}{r_{p,v}}} = 9.8577$

Min Δv for Injection: $\Delta v_{min} = v - v_{esc} = .3653$

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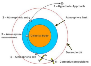

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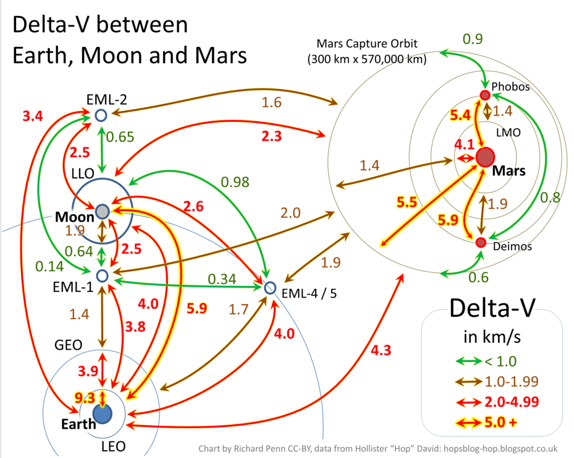
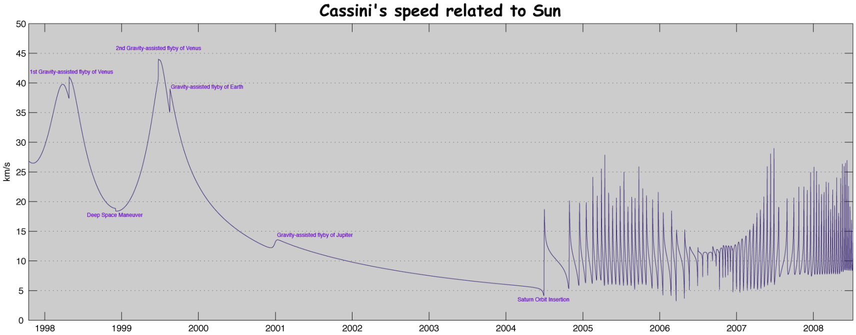
- Aerocapture is used to reduce a hyperbolic orbit to an elliptic orbit.
- Aerocapture has never been used except in Kerbel Space Program and 2010.
- Aerobraking is used to reduce the apogee of an elliptic orbit over many rotations.
- Requires a very detailed model of the atmosphere to be safe.
- Many aerobraking maneuvers are performed using Earth's atmosphere!

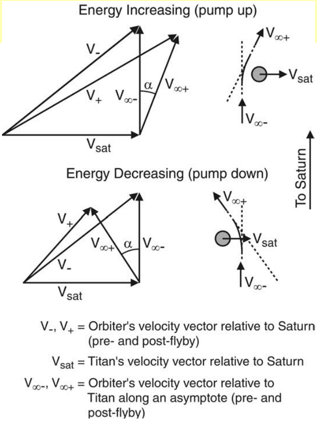
Messenger Probe to Mercury


Delta-V between
Earth, Moon and Mars
Mars Capture Orbit
(300 km x 570,000 km)
EML-2
Phobos
LMO
Mars
Deimos
LLO
Moon
EML-1
EML-4 / 5
GEO
Earth
LEO
Delta-V
in km/s
< 1.0
1.0-1.99
2.0-4.99
5.0 +
Chart by Richard Penn CC-BY, data from Hollister "Hop" David: hopsblog-hop.blogspot.co.uk



Cassini's speed related to Sun
km/s
1st Gravity-assisted flyby of Venus
2nd Gravity-assisted flyby of Venus
Gravity-assisted flyby of Earth
Deep Space Maneuver
Gravity-assisted flyby of Jupiter
Saturn Orbit Insertion
0
5
10
15
20
25
30
35
40
45
50
1998
1999
2000
2001
2002
2003
2004
2005
2006
2007
2008

Gravity Assist Trajectories



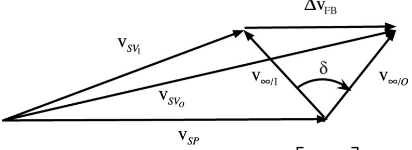
Concept: Planets rotate the relative velocity vector.

- The relative motion changes as

$$\underbrace{\vec{v}_f - \vec{v}_{planet}}_{\vec{v}_{f,rel}} = R_1(\delta)\,\underbrace{(\vec{v}_i - \vec{v}_{planet})}_{\vec{v}_{i,rel}}$$

- In the inertial frame (2 dimensions) this means

$$\vec{v}_f = \begin{bmatrix}\cos\delta & -\sin\delta \\ \sin\delta & \cos\delta\end{bmatrix}(\vec{v}_i - \vec{v}_{planet}) + \vec{v}_{planet}$$

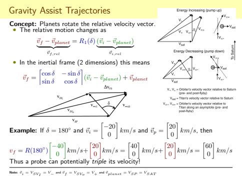


Example: If $\delta = 180°$ and $\vec{v}_i = \begin{bmatrix}-20\\0\end{bmatrix} km/s$ and $\vec{v}_p = \begin{bmatrix}20\\0\end{bmatrix} km/s$, then

$$v_f = R(180°)\begin{bmatrix}-40\\0\end{bmatrix}km/s + \begin{bmatrix}20\\0\end{bmatrix}km/s = \begin{bmatrix}40\\0\end{bmatrix}km/s + \begin{bmatrix}20\\0\end{bmatrix}km/s = \begin{bmatrix}60\\0\end{bmatrix}km/s$$

Thus a probe can potentially *triple* its velocity!

Note: $\vec{v}_i = V_{SV_I} = V_-$ and $\vec{v}_f = V_{SV_o} = V_+$ and $\vec{v}_{planet} = V_{SP} = V_{SAT}$



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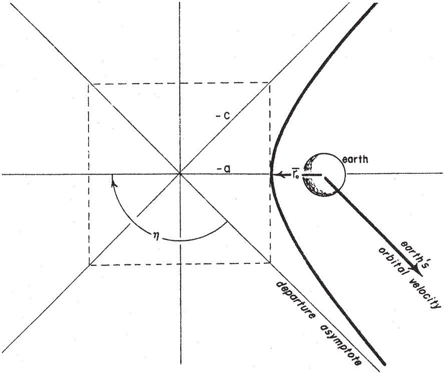
Gravity Assist Trajectories

To achieve the desired turning angle, we must control the geometry

The turning angle δ is given by

$$\delta = 2\sin^{-1}\frac{1}{e}$$



The total energy of the orbit is fixed.
Thus we can solve for

$$\boxed{a = -\mu_{planet}/\|\vec{v}_i - \vec{v}_{planet}\|^2}$$

Then the eccentricity can be fixed by the target radius as

$$\boxed{\Delta = \sqrt{\frac{a(1-e^2)\mu_{planet}}{\|\vec{v}_i - \vec{v}_{planet}\|^2}}}$$

In 3 dimensions, the calculations are more complex.

Gravity Assist Trajectories

Example: Jupiter flyby

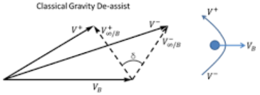
Problem: Suppose we perform a Hohman transfer from Earth to Jupiter. What is the best-case gravity assist we can expect?

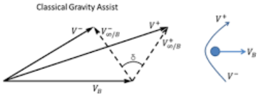
Solution: The velocity of arrival at apogee (Jupiter) in the Heliocentric frame is:

$$\vec{v}_i = v_a = \sqrt{2\mu_{sun}\frac{r_e}{r_j(r_j + r_e)}} = 7.414km/s$$

The velocity of Jupiter itself is

$$\vec{v}_{planet} = v_j = \sqrt{\frac{\mu_s}{d_j}} = 13.0573km/s$$





Since this is an outer planet, we approach from the front door. In a suitable Heliocentric frame, we have

$$\vec{v}_i = \begin{bmatrix} 7.414 \\ 0 \end{bmatrix}, \qquad \vec{v}_{planet} = \begin{bmatrix} 13.0573 \\ 0 \end{bmatrix}$$

The velocity of the spacecraft relative to Jupiter is $\vec{v}_\infty = \underbrace{\vec{v}_i - \vec{v}_p}_{\vec{v}_{i,rel}} = \begin{bmatrix} -5.6429 \\ 0 \end{bmatrix}$.

Gravity Assist Trajectories

Example: Jupiter flyby

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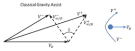

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The Heliocentric frame uses

- Jupiter Velocity vector for x axis
- Jupiter-Sun vector for y axis
- NCP for z-axis

Example: Jupiter flyby

Jupiter Data: Radius $r_j = 11.209ER$; Distance $d_j = 5.2028AU$; $\mu_j = 317.938\mu_e$.

The velocity of the spacecraft relative to jupiter is

$$\vec{v}_\infty = \vec{v}_i - \vec{v}_p = \begin{bmatrix} -5.6429 \\ 0 \end{bmatrix} km/s$$

Thus we can calculate the energy of the hyperbolic approach as

$$a = -\frac{\mu_j}{\|\vec{v}_i - \vec{v}_p\|^2} = -3.98E6km$$

The closest we can approach jupiter is its radius. If we use this for periapse, we get

$$e = 1 - \frac{r_j}{a} = 1.018$$

The eccentricity yields the maximum turning angle as

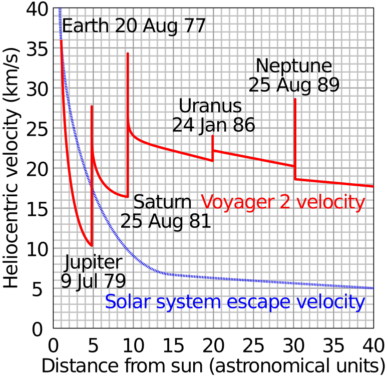
$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 158.44^\circ$$

Example: Jupiter flyby

Applying this rotation (light-side approach), we get

$$\vec{v}_f = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} (\vec{v}_i - v_{planet}) + \vec{v}_{planet} = \begin{bmatrix} 18.305 \\ -2.076 \end{bmatrix}$$

The magnitude of



Example: Jupiter flyby

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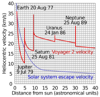
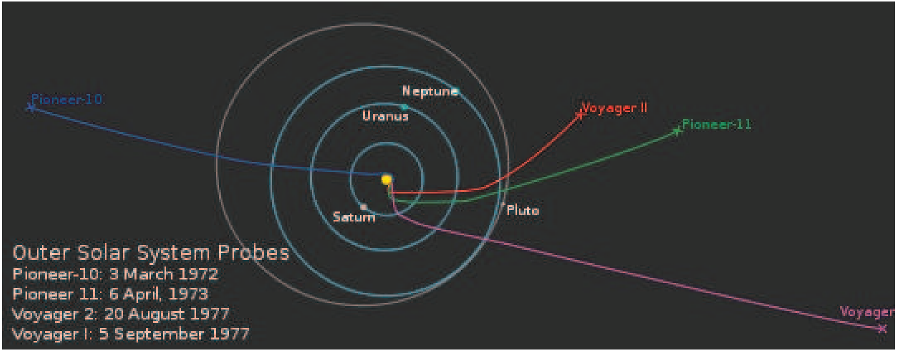


Note that if we could have reversed our direction of flight (clockwise approach), we could achieve a $\Delta v = 20.05 km/s$.

Recall the y-axis is jupiter-sun line, so the -2 component of velocity points away from sun.

Trajectories for Voyager 1, Voyager 2, and Pioneer Spacecraft



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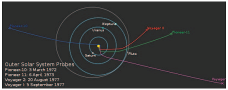

Image credit (previous page): By Cmglee

Image credit: By Phoenix7777

Summary

This Lecture you have learned:

Sphere of Influence

- Definition

Escape and Re-insertion

- The light and dark of the Oberth Effect

Patched Conics

- Heliocentric Hohmann

Planetary Flyby

- The Gravity Assist