Spacecraft Dynamics and Control

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Lecture 16: Euler's Equations

Attitude Dynamics

In this Lecture we will cover:

The Problem of Attitude Stabilization

- Actuators

Newton's Laws

- $\sum \vec{M}_i = \frac{d}{dt}\vec{H}$
- $\sum \vec{F}_i = m\frac{d}{dt}\vec{v}$

Rotating Frames of Reference

- Equations of Motion in Body-Fixed Frame
- Often Confusing

If in doubt, use the right-hand rules.

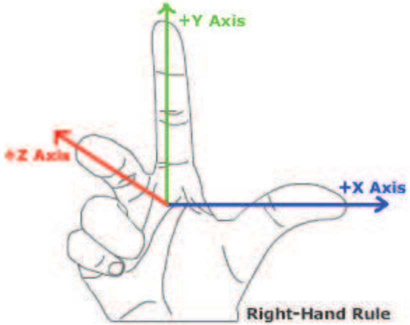


Figure: Positive Directions

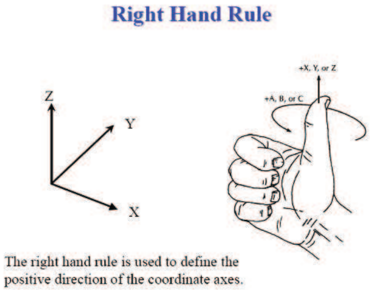
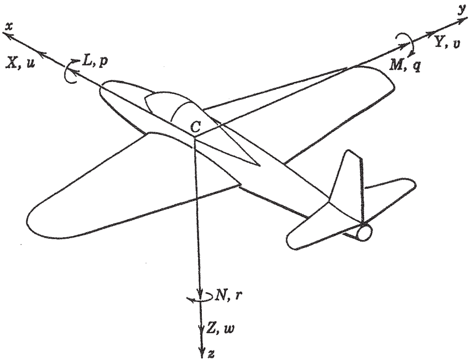


Figure: Positive Rotations

Review: Coordinate Rotations
Roll-Pitch-Yaw



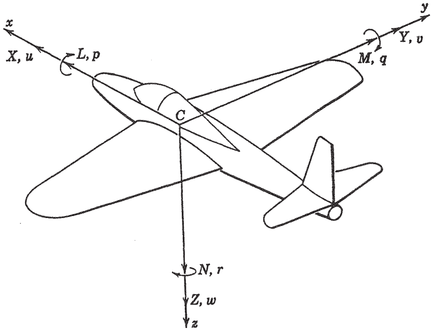
There are 3 basic rotations a vehicle can make:

- Roll = Rotation about x-axis
- Pitch = Rotation about y-axis
- Yaw = Rotation about z-axis
- Each rotation is a one-dimensional transformation.

Any two coordinate systems can be related by a sequence of 3 rotations.

Review: Forces and Moments

Forces

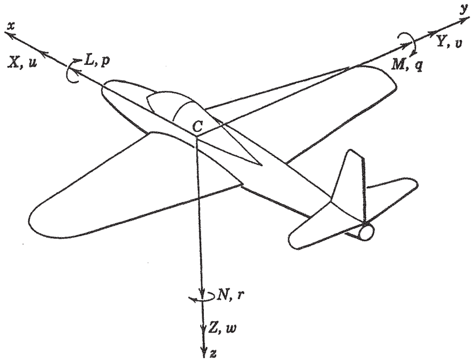


These forces and moments have standard labels. The Forces are:

X	Axial Force	Net Force in the positive x-direction
Y	Side Force	Net Force in the positive y-direction
Z	Normal Force	Net Force in the positive z-direction

Review: Forces and Moments

Moments



The Moments are called, intuitively:

L	Rolling Moment	Net Moment in the positive ω_x-direction
M	Pitching Moment	Net Moment in the positive ω_y-direction
N	Yawing Moment	Net Moment in the positive ω_z-direction

6DOF: Newton's Laws

Forces

Newton's Second Law tells us that for a particle $F = ma$. In vector form:

$$\vec{F} = \sum_i \vec{F}_i = m\frac{d}{dt}\vec{V}$$

That is, if $\vec{F} = [F_x\ F_y\ F_z]$ and $\vec{V} = [u\ v\ w]$, then

$$F_x = m\frac{du}{dt} \qquad F_x = m\frac{dv}{dt} \qquad F_z = m\frac{dw}{dt}$$

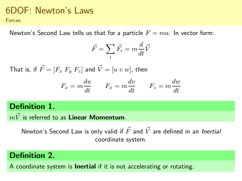
Definition 1.

$m\vec{V}$ is referred to as **Linear Momentum**.

Newton's Second Law is only valid if $\vec{F}$ and $\vec{V}$ are defined in an *Inertial* coordinate system.

Definition 2.

A coordinate system is **Inertial** if it is not accelerating or rotating.



We are not in an inertial frame because the Earth is rotating

- ECEF vs. ECI

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

Definition 3.

Where $\vec{H} = \int (\vec{r}_c \times \vec{v}_c) dm$ is the **angular momentum**.

Angular momentum of a rigid body can be found as

$$\vec{H} = I\vec{\omega}_I$$

where $\vec{\omega}_I = [p,\ q,\ r]^T$ is the angular rotation vector of the body about the center of mass.

- $p = \omega_x$ is rotation about the x-axis.
- $q = \omega_y$ is rotation about the y-axis.
- $r = \omega_z$ is rotation about the z-axis.
- ω_I is defined in an *Inertial* Frame.

The matrix I is the *Moment of Inertia Matrix* (Here also in inertial frame!).

Newton's Laws
Moments

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The matrix I is the *Moment of Inertia Matrix* (Here also in inertial frame!).

$\vec{r}_c$ and $\vec{v}_c$ are position and velocity vectors with respect to the centroid of the body.

The moment of inertia matrix is defined as

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$I_{xy} = I_{yx} = \int\int\int xydm \qquad I_{xx} = \int\int\int \left(y^2 + z^2\right) dm$$

$$I_{xz} = I_{zx} = \int\int\int xzdm \qquad I_{yy} = \int\int\int \left(x^2 + z^2\right) dm$$

$$I_{yz} = I_{zy} = \int\int\int yzdm \qquad I_{zz} = \int\int\int \left(x^2 + y^2\right) dm$$

So

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_I \\ q_I \\ r_I \end{bmatrix}$$

where p_I, q_I and r_I are the rotation vectors as expressed in the inertial frame corresponding to x-y-z.

Newton's Laws
Moment of Inertia

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So

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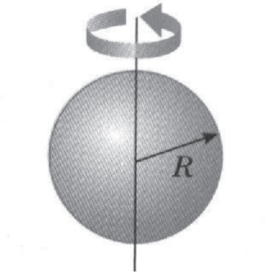
where p_I, q_I and r_I are the rotation vectors as expressed in the inertial frame corresponding to x-y-z.

- If you have symmetry about the x-y plane, $I_{xz} = I_{yz} = 0$.
- If you have symmetry about the x-z plane, $I_{xy} = I_{yz} = 0$.
- If you have symmetry about the y-z plane, $I_{xy} = I_{xz} = 0$.

- If mass is close to the x - axis plane, I_{xx} is small.
- If mass is close to the y - axis plane, I_{yy} is small.
- If mass is close to the z - axis plane, I_{zz} is small.

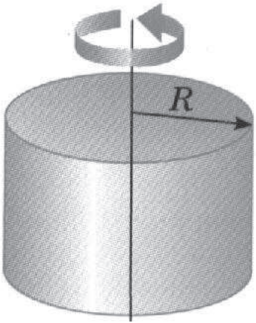


Homogeneous Sphere

$$I_{sphere} = \frac{2}{5} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ring

$$I_{ring} = m r^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogeneous Disk

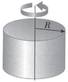
$$I_{disk} = \frac{1}{4}mr^2 \begin{bmatrix} 1+\frac{1}{3}\frac{h^2}{r^2} & 0 & 0 \\ 0 & 1+\frac{1}{3}\frac{h^2}{r^2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

F/A-18

$$I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} kslug-ft^2$$

Moment of Inertia

Examples:



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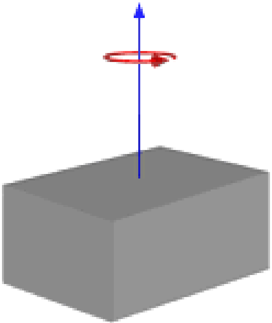
$$I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} kslug - ft^2$$

- h is the height of the disk

Moment of Inertia

Examples:



Cube

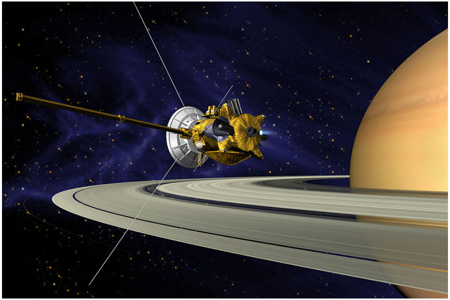
$$I_{cube} = \frac{2}{3} l^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

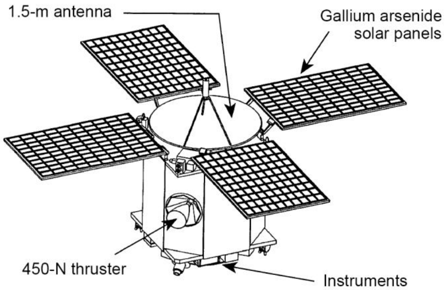
Box

$$I_{box} = \begin{bmatrix} \frac{b^2+c^2}{3} & 0 & 0 \\ 0 & \frac{a^2+c^2}{3} & 0 \\ 0 & 0 & \frac{a^2+b^2}{3} \end{bmatrix}$$

Moment of Inertia

Examples:





Cassini

$$I = \begin{bmatrix} 8655.2 & -144 & 132.1 \\ -144 & 7922.7 & 192.1 \\ 132.1 & 192.1 & 4586.2 \end{bmatrix} kg \cdot m^2$$

NEAR Shoemaker

$$I = \begin{bmatrix} 473.924 & 0 & 0 \\ 0 & 494.973 & 0 \\ 0 & 0 & 269.83 \end{bmatrix} kg{\cdot}m^2$$

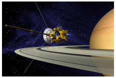
Lecture 16

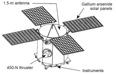
2022-04-26

Moment of Inertia

Moment of Inertia

Examples:





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NEAR Shoemaker landed on Eros in 2001

Problem:

The Body-Fixed Frame

The moment of inertia matrix, I, is fixed in the body-fixed frame. However, Newton's law only applies for an inertial frame:

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

Transport Theorem: Suppose the body-fixed frame is rotating with angular velocity vector $\vec{\omega}$. Then for any vector, $\vec{a}$, $\frac{d}{dt}\vec{a}$ in the inertial frame is

$$\frac{d\vec{a}}{dt}\Big|_I = \frac{d\vec{a}}{dt}\Big|_B + \vec{\omega} \times \vec{a}$$

Specifically, for Newton's Second Law

$$\vec{F} = m\frac{d\vec{V}}{dt}\Big|_B + m\vec{\omega} \times \vec{V}$$

and

$$\vec{M} = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

Equations of Motion

Displacement

The equation for acceleration (which we will ignore) is:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m\frac{d\vec{V}}{dt}\Big|_B + m\vec{\omega} \times \vec{V}$$

$$= m\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + m\det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_x & \omega_y & \omega_z \\ u & v & w \end{bmatrix}$$

$$= m\begin{bmatrix} \dot{u} + \omega_y w - \omega_z v \\ \dot{v} + \omega_z u - \omega_x w \\ \dot{w} + \omega_x v - \omega_y u \end{bmatrix}$$

As we will see, displacement and rotation in space are **decoupled**.

- These are the "kinematics"
- The dynamics of $\dot{\omega}$ do not depend on u, v, w.
- no aerodynamic forces (which would cause linear motion to affect rotation e.g. C_m).

Equations of Motion

The equations for rotation are:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \frac{d\vec{H}}{dt}\Big|_I = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

$$= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \vec{\omega} \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z \\ -I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y - I_{yz}\dot{\omega}_z \\ -I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix} + \vec{\omega} \times \begin{bmatrix} \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz} \\ -\omega_x I_{xy} + \omega_y I_{yy} - \omega_z I_{yz} \\ -\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z + \omega_y(\omega_z I_{zz} - \omega_x I_{xz} - \omega_y I_{yz}) - \omega_z(\omega_y I_{yy} - \omega_x I_{xy} - \omega_z I_{yz}) \\ I_{yy}\dot{\omega}_y - I_{xy}\dot{\omega}_x - I_{yz}\dot{\omega}_z - \omega_x(\omega_z I_{zz} - \omega_y I_{yz} - \omega_x I_{xz}) + \omega_z(\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}) \\ I_{zz}\dot{\omega}_z - I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + \omega_x(\omega_y I_{yy} - \omega_x I_{xy} - \omega_z I_{yz}) - \omega_y(\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}) \end{bmatrix}$$

Which is too much for any mortal. We simplify as:

- For spacecraft, we have $I_{yz} = I_{xy} = I_{xz} = 0$ (**two planes of symmetry**).
- For aircraft, we have $I_{yz} = I_{xy} = 0$ (**one plane of symmetry**).

Equations of Motion

The equations for rotation are:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \frac{d\vec{H}}{dt}\Big|_I = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

$$= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \vec{\omega} \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

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If we use the matrix version of the cross-product, we can write

$$\vec{M} = I\dot{\omega}(t) + [\omega(t)]_\times I\omega(t)$$

Which is a much-simplified version of the dynamics!
Recall

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

With $I_{xy} = I_{yz} = I_{xz} = 0$, we get: **Euler's Equations**

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy}) \\ I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) \\ I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx}) \end{bmatrix}$$

Thus:

- Rotational variables $(\omega_x, \omega_y, \omega_z)$ do not depend on translational variables (u,v,w).
  - For spacecraft, Moment forces (L,M,N) do not depend on rotational and translational variables.
  - Can be decoupled
- However, translational variables (u,v,w) depend on rotation $(\omega_x, \omega_y, \omega_z)$.
  - But we don't care.
  - These are the kinematics.

Euler Equations

Torque-Free Motion

Notice that even in the absence of external moments, the dynamics are still active:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x + \omega_y\omega_z(I_z - I_y) \\ I_y\dot{\omega}_y + \omega_x\omega_z(I_x - I_z) \\ I_z\dot{\omega}_z + \omega_x\omega_y(I_y - I_x) \end{bmatrix}$$

which yield the 3-state nonlinear ODE:

$$\dot{\omega}_x = -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t)$$

$$\dot{\omega}_y = -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t)$$

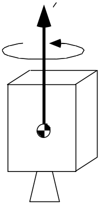
$$\dot{\omega}_z = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t)$$

Thus even in the absence of external moments

- The axis of rotation $\vec{\omega}$ will evolve
- Although the angular momentum vector $\vec{h}$ will NOT.
  - occurs because tensor I changes in inertial frame.
- This can be problematic for spin-stabilization!

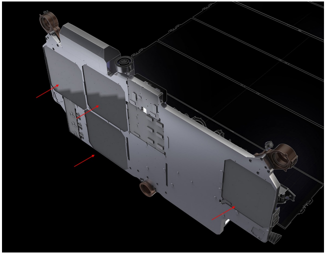
We can use Euler's equation to study Spin Stabilization.

There are two important cases:



Axisymmetric: $I_x = I_y$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{bmatrix}$$



Non-Axisymmetric: $I_x \neq I_y$

$$I = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} kg \cdot m^2$$

Rough Estimate w/o solar panel

- real data not available

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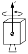

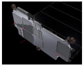

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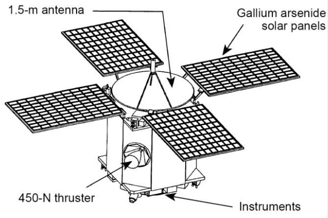
Note we say a body is axisymmetric if $I_x = I_y$.

- We don't need rotational symmetry...



Non-Axisymmetric: $I_x \neq I_y$

$$I = \begin{bmatrix} 473.924 & 0 & 0 \\ 0 & 494.973 & 0 \\ 0 & 0 & 269.83 \end{bmatrix} kg \cdot m^2$$

Spin Stabilization

Axisymmetric Case

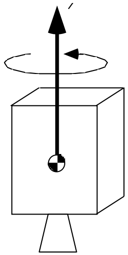
An important case is spin-stabilization of an axisymmetric spacecraft.

- Assume symmetry about z-axis ($I_x = I_y$)

Then recall

$$\dot{\omega}_z(t) = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t) = 0$$

Thus $\omega_z = constant$.

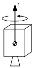


The equations for ω_x and ω_y are now

$$\begin{bmatrix}\dot{\omega}_x(t)\\ \dot{\omega}_y(t)\end{bmatrix} = \begin{bmatrix} 0 & -\frac{I_z - I_y}{I_x}\omega_z \\ -\frac{I_x - I_z}{I_y}\omega_z & 0\end{bmatrix}\begin{bmatrix}\omega_x(t)\\ \omega_y(t)\end{bmatrix}$$

Which is a linear ODE.

Spin Stabilization
Axisymmetric Case



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Which is a linear ODE.

ONLY FOR THE AXISYMMETRIC CASE!!!!!!!!!!!!!!!!!!!!!!!!!!

$$I_x = I_y$$

Spin Stabilization
Axisymmetric Case

Fortunately, linear systems have closed-form solutions.
let $\lambda = \frac{I_z - I_x}{I_x}\omega_z$. Then

$$\dot{\omega}_x(t) = -\lambda\omega_y(t)$$
$$\dot{\omega}_y(t) = \lambda\omega_x(t)$$

Combining, we get

$$\ddot{\omega}_x(t) = -\lambda^2\omega_x(t)$$

which has solution

$$\omega_x(t) = \omega_x(0)\cos(\lambda t) + \frac{\dot{\omega}_x(0)}{\lambda}\sin(\lambda t)$$

Differentiating, we get

$$\begin{aligned}\omega_y(t) &= -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0)\sin(\lambda t) - \frac{\dot{\omega}_x(0)}{\lambda}\cos(\lambda t)\\ &= \omega_x(0)\sin(\lambda t) + \omega_y(0)\cos(\lambda t)\\ \omega_x(t) &= \omega_x(0)\cos(\lambda t) - \omega_y(0)\sin(\lambda t)\end{aligned}$$

Spin Stabilization

Axisymmetric Case

Define $\omega_{xy} = \sqrt{\omega_x^2 + \omega_y^2}$.

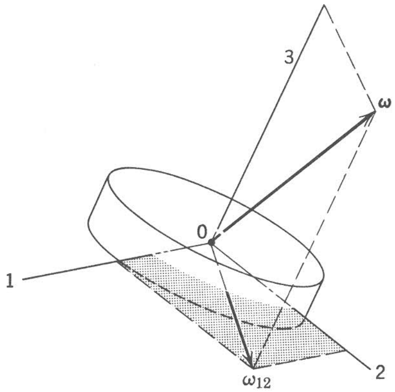
$$
\begin{aligned}
\omega_{xy}^2 &= (\omega_x(0)\sin(\lambda t) + \omega_y(0)\cos(\lambda t))^2 + (\omega_x(0)\cos(\lambda t) - \omega_y(0)\sin(\lambda t))^2 \\
&= \omega_x(0)^2\sin^2(\lambda t) + \omega_y(0)^2\cos^2(\lambda t) + 2\omega_x(0)\omega_y(0)\cos(\lambda t)\sin(\lambda t) \\
&\qquad + \omega_x(0)^2\cos^2(\lambda t) + \omega_y(0)^2\sin^2(\lambda t) - 2\omega_x(0)\omega_y(0)\cos(\lambda t)\sin(\lambda t) \\
&= \omega_x(0)^2(\sin^2(\lambda t) + \cos^2(\lambda t)) + \omega_y(0)^2(\cos^2(\lambda t) + \sin^2(\lambda t)) \\
&= \omega_x(0)^2 + \omega_y(0)^2
\end{aligned}
$$

Thus

- ω_z is constant
  - rotation about axis of symmetry
- $\sqrt{\omega_x^2 + \omega_y^2}$ is constant
  - rotation perpendicular to axis of symmetry

This type of motion is often called **Precession**!

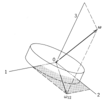
Circular Motion in the Body-Fixed Frame



Thus

$$\omega(t) = \begin{bmatrix}\omega_x(t)\\ \omega_y(t)\\ \omega_z(t)\end{bmatrix} = \begin{bmatrix}\cos(\lambda t) & -\sin(\lambda t) & 0\\ \sin(\lambda t) & \cos(\lambda t) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\omega_x(0)\\ \omega_y(0)\\ \omega_z(0)\end{bmatrix} = R_3(\lambda t)\begin{bmatrix}\omega_x(0)\\ \omega_y(0)\\ \omega_z(0)\end{bmatrix}$$

Circular Motion in the Body-Fixed Frame



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- For $\lambda > 0$, this is a Positive (counterclockwise) rotation, about the z-axis, of the angular velocity vector ω *as expressed in the body-fixed coordinates!*

Prolate vs. Oblate

The speed of the precession is given by the natural frequency:

$$\lambda = \frac{I_z - I_x}{I_x}\omega_z$$

with period $T = \frac{2\pi}{\lambda} = \frac{2\pi I_x}{I_z - I_x}\omega_z^{-1}$.

Direction of Precession: There are two cases

Definition 4 (Direct).

An axisymmetric (about z-axis) rigid body is **Prolate** if $I_z < I_x = I_y$.

Definition 5 (Retrograde).

An axisymmetric (about z-axis) rigid body is **Oblate** if $I_z > I_x = I_y$.

Thus we have two cases:

- $\lambda > 0$ if object is *Oblate* (CCW rotation)
- $\lambda < 0$ if object is *Prolate* (CW rotation)

Note that these are rotations of ω, as expressed in the **Body-Fixed** Frame.

Pay Attention to the Body-Fixed Axes

Figure: Prolate Precession

Figure: Oblate Precession

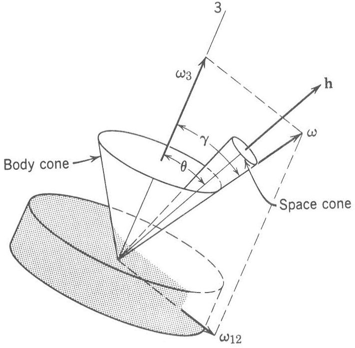
The black arrow is $\vec{\omega}$.

- The body-fixed x and y axes are indicated with red and green dots.
- Notice the direction of rotation of ω with respect to these dots.
- The angular momentum vector is the inertial z axis.

Motion in the Inertial Frame

As these videos illustrate, we are typically interested in motion in the **Inertial Frame**.

- Use of Rotation Matrices is complicated.
  - Which coordinate system to use???
- Lets consider motion relative to $\vec{h}$.
  - Which is fixed in inertial space.



We know that in Body-Fixed coordinates,

$$\vec{h} = I\vec{\omega} = \begin{bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{bmatrix}$$

Now lets find the orientation of ω and $\hat{z}$ with respect to this fixed vector.

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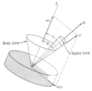


Now lets find the orientation of ω and $\hat{z}$ with respect to this fixed vector.

- The “Space Cone” is how ω moves in inertial coordinates
- The “Body Cone” is how ω moves with respect to the body.

Motion in the Inertial Frame

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ define the body-fixed unit vectors.

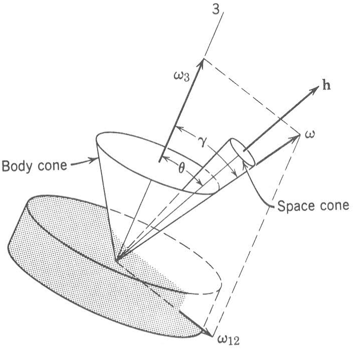
We first note that since $I_x = I_y$ and

$$\begin{aligned}\vec{h} &= I_x\omega_x\hat{x} + I_y\omega_y\hat{y} + I_z\omega_z\hat{z} \\ &= I_x(\omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}) + (I_z - I_x)\omega_z\hat{z} \\ &= I_x\vec{\omega} + (I_z - I_x)\omega_z\hat{z}\end{aligned}$$

we have that

$$\vec{\omega} = \frac{1}{I_x}\vec{h} + \frac{I_x - I_z}{I_x\omega_z}\hat{z}$$

which implies that $\vec{\omega}$ lies in the $\hat{z} - \vec{h}$ plane.



Motion in the Inertial Frame

We now focus on two constants of motion

- θ - The angle $\vec{h}$ makes with the body-fixed $\hat{z}$ axis.
- γ - The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

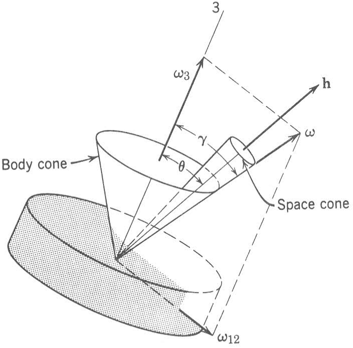
Since

$$\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{bmatrix}$$

The angle θ is defined by

$$\tan\theta = \frac{\sqrt{h_x^2 + h_y^2}}{h_z} = \frac{I_x\sqrt{\omega_x^2 + \omega_y^2}}{I_z\omega_z} = \frac{I_x}{I_z}\frac{\omega_{xy}}{\omega_z}$$

Since ω_{xy} and ω_z are fixed, θ is a constant of motion.



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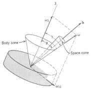


Again, $\vec{h}$ here is in the body-fixed frame

- This is why it changes over time.

Motion in the Inertial Frame

The second angle to consider is

- γ - The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

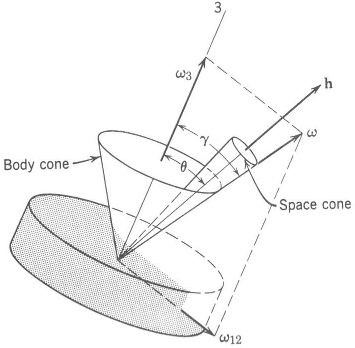
As before, the angle γ is defined by

$$\tan\gamma = \frac{\sqrt{\omega_x^2 + \omega_y^2}}{\omega_z} = \frac{\omega_{xy}}{\omega_z}$$

Since ω_{xy} and ω_z are fixed, γ is a constant of motion.

- We have the relationship

$$\tan\theta = \frac{I_x}{I_z}\frac{\omega_{xy}}{\omega_z} = \frac{I_x}{I_z}\tan\gamma$$



Thus we have two cases:

1. $I_x > I_z$ - Then $\theta > \gamma$
2. $I_x < I_z$ - Then $\theta < \gamma$ (As Illustrated)

Motion in the Inertial Frame

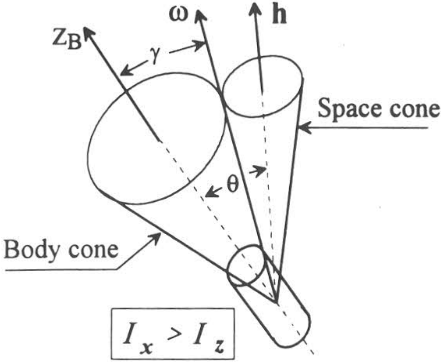


Figure: The case of $I_x > I_z$ $(\theta > \gamma)$

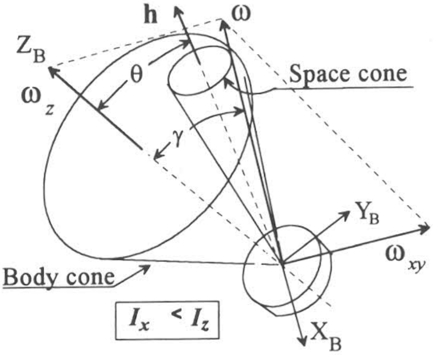


Figure: The case of $I_z > I_x$ $(\gamma > \theta)$

Motion in the Inertial Frame

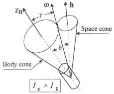


Figure: The case of $I_x > I_z$ ($\theta > \gamma$)

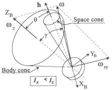


Figure: The case of $I_z > I_x$ ($\gamma > \theta$)

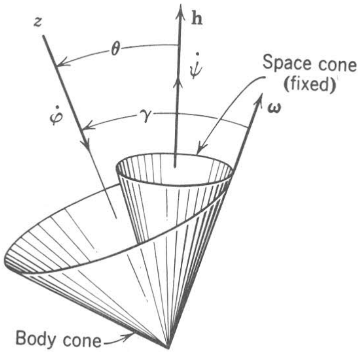
We illustrate the motion using the Space Cone and Body Cone

- The space cone is fixed in inertial space (doesn't move)
- The space cone has width $|\omega - \theta|$
- The body cone is centered around the z-axis of the body.
- In body-fixed coordinates, the space cone rolls around the body cone (which is fixed)
- In inertial coordinates, the body cone rolls around the space cone (which is fixed)

Motion in the Inertial Frame

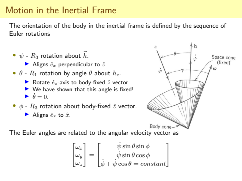
The orientation of the body in the inertial frame is defined by the sequence of Euler rotations

- ψ - R_3 rotation about $\vec{h}$.
  - Aligns $\hat{e}_x$ perpendicular to $\hat{z}$.
- θ - R_1 rotation by angle θ about h_x.
  - Rotate $\hat{e}_z$-axis to body-fixed $\hat{z}$ vector
  - We have shown that this angle is fixed!
  - $\dot{\theta} = 0$.
- ϕ - R_3 rotation about body-fixed $\hat{z}$ vector.
  - Aligns $\hat{e}_x$ to $\hat{x}$.



The Euler angles are related to the angular velocity vector as

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\psi}\sin\theta\sin\phi \\ \dot{\psi}\sin\theta\cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta = constant \end{bmatrix}$$



This comes from

$$\vec{\omega} = R_3(\phi)R_1(\theta)R_3(\psi)\begin{bmatrix}0\\0\\\dot{\psi}\end{bmatrix} + R_3(\phi)R_1(\theta)\begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix} + R_3(\phi)\begin{bmatrix}0\\0\\\dot{\phi}\end{bmatrix}$$

$$= \begin{bmatrix}\dot{\psi}\sin\theta\sin\phi\\\dot{\psi}\sin\theta\cos\phi\\\dot{\psi}\cos\theta\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\phi}\end{bmatrix}$$

Motion in the Inertial Frame

To find the motion of ω, we differentiate

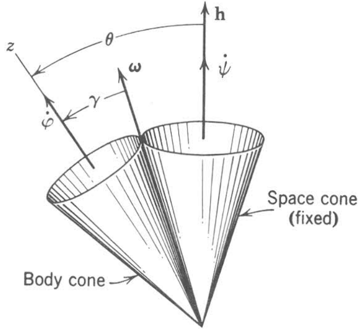
$$\begin{bmatrix}\dot\omega_x\\ \dot\omega_y\\ \dot\omega_z\end{bmatrix} = \begin{bmatrix}\dot\psi\dot\phi\sin\theta\cos\phi\\ -\dot\psi\dot\phi\sin\theta\sin\phi\\ 0\end{bmatrix}$$

Now, substituting into the Euler equations yields

$$\dot\psi = \frac{I_z}{(I_x - I_z)\cos\theta}\dot\phi$$

There are two cases here:

- $I_x > I_z$ - **Direct** precession
  - $\dot\psi$ and $\dot\phi$ aligned.
- $I_y > I_x$ - **Retrograde** precession
  - $\dot\psi$ and $\dot\phi$ are opposite.



Motion in the Inertial Frame

To find the motion of ω, we differentiate

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Now, substituting into the Euler equations yields

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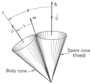


Recall $\dot{\omega}_x$ and $\dot{\omega}_y$ can be expressed in terms of ω_x and ω_y

Motion in the Inertial Frame

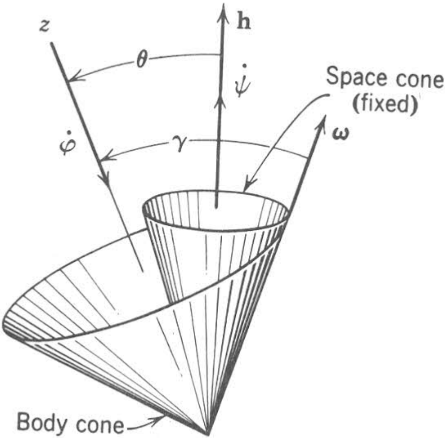


Figure: Retrograde Precession ($I_z > I_x$)

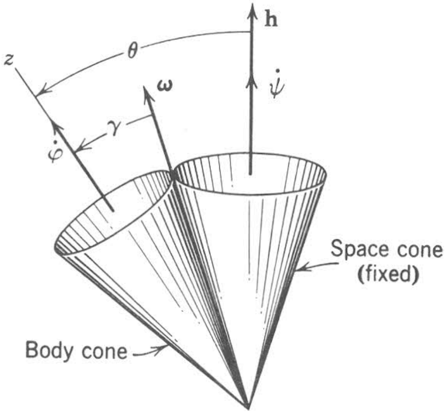


Figure: Direct Precession ($I_z < I_x$)

Mathematica Demonstrations

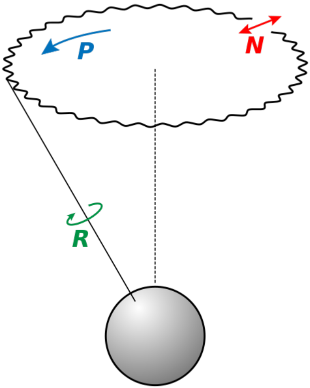
Mathematica Precession Demonstration

Prolate and Oblate Spinning Objects

Figure: Prolate Object: $I_x = I_y = 4$ and $I_z = 1$

Figure: Oblate Object: Vesta

Next Lecture



Note Bene: Precession of a spacecraft is often called nutation (θ is called the nutation angle).

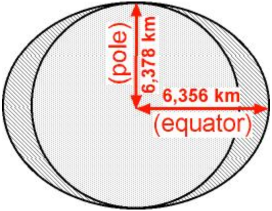
- By most common definitions, for torque-free motions, $N = 0$
  - Free rotation has NO nutation.
  - This is confusing

Problem: The earth is 42.72 km wider than it is tall. How quickly will the rotational axis of the earth precess due to this effect?

Solution: for an axisymmetric ellipsoid with height a and width b, we have $I_x = I_y = \frac{1}{5}m(a^2 + b^2)$ and $I_z = \frac{2}{5}mb^2$.

Thus $b = 6378km$, $a = 6352km$ and we have $(m_e = 5.974 \cdot 10^{24} kg)$

$$I_z = 9.68 \cdot 10^{37kg-m^2}, \qquad I_x = I_y = 9.72 \cdot 10^{37kg-m^2}$$



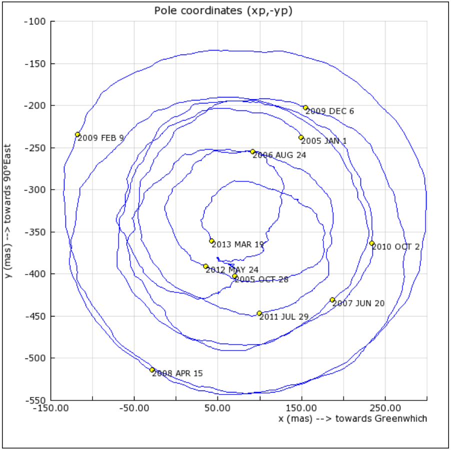
If we take $\omega_z = \frac{2\pi}{T} \cong 2\pi day^{-1}$, then we have

$$\lambda = \frac{I_z - I_x}{I_x}\omega_z = .0041 day^{-1}$$

That gives a period of $T = \frac{2\pi}{\lambda} = 243.5 days$. This motion of the earth is known as the **Chandler Wobble**.

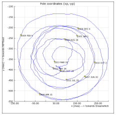
Note: This is only the Torque-free precession.

Precession



- Actual period is 434 days
  - Actual $I_x = I_y = 8.008 \cdot 10^{37} kg - m^2$.
  - Actual $I_z = 8.034 \cdot 10^{37} kg - m^2$.
  - Which would predict $T = 306$days

Precession



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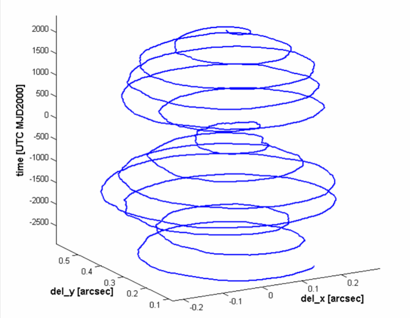
  - Actual $I_z = 8.034 \cdot 10^{37} kg - m^2$.
  - Which would predict $T = 306$days

- The precession of the earth was first notices by Euler, D'Alembert and Lagrange as slight variations in lattitude.
- Error partially due to fact Earth is not a rigid body(Chandler + Newcomb).
- Magnitude of around 9m
- Previous plot scale is milli-arc-seconds (mas)



Next Lecture

In the next lecture we will cover

Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation