Spacecraft Dynamics and Control

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Lecture 16: Euler’s Equations

1. Euler’s Equations.
2. Applications to Body Cones.
Attitude Dynamics

In this Lecture we will cover:

The Problem of Attitude Stabilization

- Actuators

Newton’s Laws

- \[ \sum M_i \dot{H} = \frac{d}{dt} \vec{H} \]
- \[ \sum F_i = m \frac{d}{dt} \vec{v} \]

Rotating Frames of Reference

- Equations of Motion in Body-Fixed Frame
- Often Confusing
If in doubt, use the right-hand rules.
There are 3 basic rotations a vehicle can make:

- Roll = Rotation about $x$-axis
- Pitch = Rotation about $y$-axis
- Yaw = Rotation about $z$-axis

Each rotation is a one-dimensional transformation.

Any two coordinate systems can be related by a sequence of 3 rotations.
Review: Forces and Moments

Forces

These forces and moments have standard labels. The Forces are:

- **X**: Axial Force, Net Force in the positive $x$-direction
- **Y**: Side Force, Net Force in the positive $y$-direction
- **Z**: Normal Force, Net Force in the positive $z$-direction
The Moments are called, intuitively:

- **L**: Rolling Moment, Net Moment in the positive $\omega_x$-direction
- **M**: Pitching Moment, Net Moment in the positive $\omega_y$-direction
- **N**: Yawing Moment, Net Moment in the positive $\omega_z$-direction
Newton’s Second Law tells us that for a particle, \( F = ma \). In vector form:

\[
\vec{F} = \sum_i \vec{F}_i = m \frac{d\vec{V}}{dt}
\]

That is, if \( \vec{F} = [F_x \ F_y \ F_z] \) and \( \vec{V} = [u \ v \ w] \), then

\[
F_x = m \frac{du}{dt} \quad F_y = m \frac{dv}{dt} \quad F_z = m \frac{dw}{dt}
\]

**Definition 1.**

\( m\vec{V} \) is referred to as **Linear Momentum**.

Newton’s Second Law is only valid if \( \vec{F} \) and \( \vec{V} \) are defined in an **Inertial** coordinate system.

**Definition 2.**

A coordinate system is **Inertial** if it is not accelerating or rotating.
Newton’s Second Law tells us that for a particle \( F = ma \). In vector form:

\[
\vec{F} = \sum_i \vec{F}_i = m \frac{d\vec{V}}{dt}
\]

That is, if \( \vec{F} = [F_x, F_y, F_z] \) and \( \vec{V} = [v_x, v_y, v_z] \), then

\[
F_x = m \frac{dv_x}{dt} \quad F_y = m \frac{dv_y}{dt} \quad F_z = m \frac{dv_z}{dt}
\]

**Definition 1.**

\( m\vec{V} \) is referred to as **Linear Momentum**.

Newton’s Second Law is only valid if \( \vec{F} \) and \( \vec{V} \) are defined in an **Inertial** coordinate system.

**Definition 2.**

A coordinate system is **Inertial** if it is not accelerating or rotating.

We are not in an inertial frame because the Earth is rotating.

- **ECEF** vs. **ECI**

  "body-fixed" vs. "inertial"
Newton’s Laws

Moments

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

\[ \vec{M} = \sum_i \vec{M}_i = \frac{d}{dt} \vec{H} \]

**Definition 3.**

Where \( \vec{H} = \int (\vec{r}_c \times \vec{v}_c) dm \) is the **angular momentum**.

Angular momentum of a rigid body can be found as

\[ \vec{H} = I \vec{\omega}_I \]

where \( \vec{\omega}_I = [p, q, r]^T \) is the angular rotation vector of the body about the center of mass.

- \( p = \omega_x \) is rotation about the \( x \)-axis.
- \( q = \omega_y \) is rotation about the \( y \)-axis.
- \( r = \omega_z \) is rotation about the \( z \)-axis.
- \( \vec{\omega}_I \) is defined in an *Inertial* Frame.

The matrix \( I \) is the *Moment of Inertia Matrix* (Here also in *inertial* frame!).
Newton's Laws

Moments

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

\[ \vec{M} = \sum_i \vec{M}_i = \frac{d}{dt} \vec{H} \]

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- \( q = \omega_y \) is rotation about the \( y \)-axis.
- \( r = \omega_z \) is rotation about the \( z \)-axis.

\( \vec{\omega}_I \) is defined in an inertial frame. The matrix \( I \) is the Moment of Inertia Matrix (Here also in inertial frame!).

\( \vec{r}_c \) and \( \vec{v}_c \) are position and velocity vectors with respect to the centroid of the body.
The moment of inertia matrix is defined as

\[
I = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\]

\[I_{xy} = I_{yx} = \int \int \int xy \, dm\]
\[I_{xz} = I_{zx} = \int \int \int xz \, dm\]
\[I_{yz} = I_{zy} = \int \int \int yz \, dm\]

\[I_{xx} = \int \int \int (y^2 + z^2) \, dm\]
\[I_{yy} = \int \int \int (x^2 + z^2) \, dm\]
\[I_{zz} = \int \int \int (x^2 + y^2) \, dm\]

So

\[
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix} = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix} \begin{bmatrix}
pI \\
qI \\
rI
\end{bmatrix}
\]

where \(p_I\), \(q_I\) and \(r_I\) are the rotation vectors as expressed in the inertial frame corresponding to \(x-y-z\).
The moment of inertia matrix is defined as

\[
I = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{xy} & I_{yy} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{zz}
\end{bmatrix}
\]

\[
I_{xy} = I_{yx} = \int \int \int xy dm \\
I_{xz} = I_{zx} = \int \int \int xz dm \\
I_{yz} = I_{zy} = \int \int \int yz dm
\]

So

\[
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix} = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{xy} & I_{yy} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{zz}
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

where \(p\), \(q\), and \(r\) are the rotation vectors as expressed in the inertial frame corresponding to \(x\)-, \(y\)-, and \(z\)-axis.

- If you have symmetry about the \(x\)-\(y\) plane, \(I_{xz} = I_{yz} = 0\).
- If you have symmetry about the \(x\)-\(z\) plane, \(I_{xy} = I_{yz} = 0\).
- If you have symmetry about the \(y\)-\(z\) plane, \(I_{xy} = I_{xz} = 0\).
- If mass is close to the \(x\) - axis plane, \(I_{xx}\) is small.
- If mass is close to the \(y\) - axis plane, \(I_{yy}\) is small.
- If mass is close to the \(z\) - axis plane, \(I_{zz}\) is small.
Moment of Inertia

Examples:

**Homogeneous Sphere**

\[
I_{sphere} = \frac{2}{5} mr^2 \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

**Ring**

\[
I_{ring} = mr^2 \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
Moment of Inertia

Examples:

Homogeneous Disk

\[
I_{\text{disk}} = \frac{1}{4}mr^2 \begin{bmatrix} 1 + \frac{1}{3} \frac{h}{r^2} & 0 & 0 \\ 0 & 1 + \frac{1}{3} \frac{h}{r^2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}
\]

F/A-18

\[
I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} \text{ kslug} - \text{ft}^2
\]
Moment of Inertia

- $h$ is the height of the disk
Moment of Inertia

Examples:

Cube

\[ I_{cube} = \frac{2}{3} l^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Box

\[ I_{box} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{b^2+c^2}{3} & 0 & 0 \\ 0 & \frac{a^2+c^2}{3} & 0 \\ 0 & 0 & \frac{a^2+b^2}{3} \end{bmatrix} \]
Moment of Inertia

Examples:

Casinni

\[
I = \begin{bmatrix}
8655.2 & -144 & 132.1 \\
-144 & 7922.7 & 192.1 \\
132.1 & 192.1 & 4586.2
\end{bmatrix} \text{ kg} \cdot \text{m}^2
\]

NEAR Shoemaker

\[
I = \begin{bmatrix}
473.924 & 0 & 0 \\
0 & 494.973 & 0 \\
0 & 0 & 269.83
\end{bmatrix} \text{ kg} \cdot \text{m}^2
\]
Moment of Inertia

Examples:

Cassini

\[
\begin{bmatrix}
8655.2 & -144.132 & 132.1 \\
-144 & 7822.7 & 102.1 \\
132.1 & 102.1 & 192.7
\end{bmatrix}
\] kg \cdot m^2

NEAR Shoemaker

\[
\begin{bmatrix}
473.124 & 0 & 0 \\
0 & 494.973 & 0 \\
0 & 0 & 209.83
\end{bmatrix}
\] kg \cdot m^2

NEAR Shoemaker landed on Eros in 2001
The moment of inertia matrix, \( I \), is fixed in the body-fixed frame. However, Newton’s law only applies for an inertial frame:

\[
\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt} \vec{H}
\]

**Transport Theorem:** Suppose the body-fixed frame is rotating with angular velocity vector \( \vec{\omega} \). Then for any vector, \( \vec{a} \), \( \frac{d}{dt} \vec{a} \) in the inertial frame is

\[
\frac{d}{dt} \vec{a} \bigg|_B = \frac{d}{dt} \vec{a} \bigg|_I + \vec{\omega} \times \vec{a}
\]

Specifically, for Newton’s Second Law

\[
\vec{F} = m \frac{d\vec{V}}{dt} \bigg|_B + m\vec{\omega} \times \vec{V}
\]

and

\[
\vec{M} = \frac{d\vec{H}}{dt} \bigg|_B + \vec{\omega} \times \vec{H}
\]
Equations of Motion

Displacement

The equation for acceleration (which we will ignore) is:

\[
\begin{bmatrix}
F_x \\
F_y \\
F_z
\end{bmatrix} = m \frac{d\vec{V}}{dt} + m\vec{\omega} \times \vec{V}
\]

\[
= m \begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix} + m \det \begin{bmatrix}
\ddot{x} & \ddot{y} & \ddot{z} \\
\omega_x & \omega_y & \omega_z \\
u & v & w
\end{bmatrix}
\]

\[\mathbf{F}_b = m \begin{bmatrix}
\dot{u} + \omega_y w - \omega_z v \\
\dot{v} + \omega_z u - \omega_x w \\
\dot{w} + \omega_x v - \omega_y u
\end{bmatrix}
\]

As we will see, displacement and rotation in space are **decoupled**.

- These are the “kinematics”
- The dynamics of \(\vec{\omega}\) do not depend on \(u, v, w\).
- **no aerodynamic forces** (which would cause linear motion to affect rotation e.g. \(C_m\)).
Equations of Motion for Rotation

The equations for rotation are:

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} = \frac{d\mathbf{H}}{dt}
\bigg|_I = \frac{d\mathbf{H}}{dt}
\bigg|_B + \mathbf{\omega} \times \mathbf{H} = I_{13} \mathbf{\ddot{\omega}}_{13} + \mathbf{\ddot{\omega}}_B \times (I_B \mathbf{\dot{\omega}}_B)
\]

\[
\begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix}
+ \mathbf{\ddot{\omega}} \times 
\begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}

\]

Which is too much for any mortal. We simplify as:

- For spacecraft, we have \( I_{yz} = I_{xy} = I_{xz} = 0 \) (two planes of symmetry).
- For aircraft, we have \( I_{yz} = I_{xy} = 0 \) (one plane of symmetry).
If we use the matrix version of the cross-product, we can write

$$\vec{M} = I\dot{\omega}(t) + [\omega(t)] \times I\omega(t)$$

Which is a much-simplified version of the dynamics!

Recall

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$
Equations of Motion

Euler Moment Equations

With \( I_{xy} = I_{yz} = I_{xz} = 0 \), we get: **Euler’s Equations**

\[
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix} = 
\begin{bmatrix}
I_{xx} \dot{\omega}_x + \omega_y \omega_z (I_{zz} - I_{yy}) \\
I_{yy} \dot{\omega}_y + \omega_x \omega_z (I_{xx} - I_{zz}) \\
I_{zz} \dot{\omega}_z + \omega_x \omega_y (I_{yy} - I_{xx})
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

Thus:

- Rotational variables \((\omega_x, \omega_y, \omega_z)\) do not depend on translational variables \((u, v, w)\).
  - For spacecraft, Moment forces \((L, M, N)\) do not depend on rotational and translational variables.
  - Can be decoupled

- However, translational variables \((u, v, w)\) depend on rotation \((\omega_x, \omega_y, \omega_z)\).
  - But we don’t care.
  - These are the kinematics.
Notice that even in the absence of external moments, the dynamics are still active:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
I_x \dot{\omega}_x + \omega_y \omega_z (I_z - I_y) \\
I_y \dot{\omega}_y + \omega_x \omega_z (I_x - I_z) \\
I_z \dot{\omega}_z + \omega_x \omega_y (I_y - I_x)
\end{bmatrix}
\]

which yield the 3-state nonlinear ODE:

\[
\begin{align*}
\dot{\omega}_x &= -\frac{I_z - I_y}{I_x} \omega_y(t)\omega_z(t) \\
\dot{\omega}_y &= -\frac{I_x - I_z}{I_y} \omega_x(t)\omega_z(t) \\
\dot{\omega}_z &= -\frac{I_y - I_x}{I_z} \omega_x(t)\omega_y(t)
\end{align*}
\]

Thus even in the absence of external moments:

- The axis of rotation $\vec{\omega}$ will evolve.
- Although the angular momentum vector $\vec{h}$ will NOT.
  - ▶ occurs because tensor $I$ changes in inertial frame.
- This can be problematic for spin-stabilization!
We can use Euler’s equation to study Spin Stabilization.

There are two important cases:

1. **Axisymmetric:** \( I_x = I_y \)
   
   \[
   I = \begin{bmatrix}
   I_x & 0 & 0 \\
   0 & I_x & 0 \\
   0 & 0 & I_z 
   \end{bmatrix}
   \]

2. **Non-Axisymmetric:** \( I_x \neq I_y \)
   
   \[
   I = \begin{bmatrix}
   10 & 0 & 0 \\
   0 & 3 & 0 \\
   0 & 0 & \text{small} 
   \end{bmatrix}
   \text{kg} \cdot \text{m}^2
   
   Rough Estimate w/o solar panel
   - real data not available
Euler Equations

Note we say a body is axisymmetric if $I_x = I_y$.

- We don’t need rotational symmetry...

Non-Axisymmetric: $I_x \neq I_y$

\[
I = \begin{bmatrix}
473.924 & 0 & 0 \\
0 & 494.973 & 0 \\
0 & 0 & 269.83
\end{bmatrix} \text{ kg} \cdot \text{m}^2
\]
An important case is spin-stabilization of an axisymmetric spacecraft.

- Assume symmetry about z-axis \((I_x = I_y)\)

Then recall

\[
\dot{\omega}_z(t) = -\frac{I_y - I_x}{I_z} \omega_x(t) \omega_y(t) = 0
\]

Thus \(\omega_z = \text{constant}\).

The equations for \(\omega_x\) and \(\omega_y\) are now

\[
\begin{bmatrix}
\dot{\omega}_x(t) \\
\dot{\omega}_y(t)
\end{bmatrix} =
\begin{bmatrix}
0 & -\frac{I_z - I_x}{I_y} \\
-\frac{I_x - I_z}{I_y} & 0
\end{bmatrix}
\begin{bmatrix}
\omega_x(t) \\
\omega_y(t)
\end{bmatrix}
\]

Which is a linear ODE.
An important case is spin stabilization of an axisymmetric spacecraft.

- Assume symmetry about z-axis ($I_x = I_y$)
- Then recall

$$\dot{\omega}_z(t) = -I_y - I_x I_z \omega_x(t) - I_x I_z \omega_y(t) = 0$$

Thus $\omega_z = \text{constant}$.

Angular Momentum coincides with Nominal Spin Axis

In the absence of energy dissipation, nutational motion is stable about the axis of either the maximum or minimum moment of inertia. This implies that the amplitude of motion is bounded by initial conditions. However, all real spacecraft experience some form of energy dissipation. In this case, nutational motion is only stable about the axis of maximum moment of inertia. The axes of minimum and maximum moments of inertia are referred to as minor and major axes, respectively. Thus, if a spacecraft is spinning about its minor axis, nutational motion will grow until the spacecraft tumbles and eventually reorients itself spinning about its major axis. Reorientation of the spin axis is illustrated in Figure 5. Conversely, if a spacecraft is spinning about its major axis, any nutational motion will simply decay.

ASMOS can be used to investigate stability and energy dissipation effects. With ASMOS, the user can introduce various rates of internal energy dissipation into the rigid body model by entering viscous damping coefficients and wheel inertias. The user can then watch resulting motion. This motion can also be plotted for further analysis.

Conclusion

ASMOS is a simulation tool that incorporates animated 3-D computer graphics to visualize spacecraft attitude motion. The program runs on Macintosh personal computers and features pull down menus and dialog boxes making the program accessible and easy to use. The program is capable of simulating and animating a wide range of rigid body attitude motion. The rigid body model includes an energy sink for investigating stability and energy dissipation effects.

References

Spin Stabilization
Axisymmetric Case

Fortunately, linear systems have closed-form solutions.

Let \( \lambda = \frac{I_z - I_x}{I_x} \omega_z \). Then

\[
\begin{align*}
\dot{\omega}_x(t) &= -\lambda \omega_y(t) \\
\dot{\omega}_y(t) &= \lambda \omega_x(t)
\end{align*}
\]

Combining, we get

\[
\dot{\omega}_x(t) = -\lambda^2 \omega_x(t)
\]

which has solution

\[
\omega_x(t) = \omega_x(0) \cos(\lambda t) + \frac{\dot{\omega}_x(0)}{\lambda} \sin(\lambda t)
\]

Differentiating, we get

\[
\begin{align*}
\omega_y(t) &= -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0) \sin(\lambda t) - \frac{\dot{\omega}_x(0)}{\lambda} \cos(\lambda t) \\
\omega_x(t) &= \omega_x(0) \cos(\lambda t) + \omega_y(0) \cos(\lambda t)
\end{align*}
\]
Define $\omega_{xy} = \sqrt{\omega_x^2 + \omega_y^2}$.

\[
\begin{align*}
\omega_{xy}^2 &= (\omega_x(0) \sin(\lambda t) + \omega_y(0) \cos(\lambda t))^2 + (\omega_x(0) \cos(\lambda t) - \omega_y(0) \sin(\lambda t))^2 \\
&= \omega_x(0)^2 \sin^2(\lambda t) + \omega_y(0)^2 \cos^2(\lambda t) + 2\omega_x(0)\omega_y(0) \cos(\lambda t) \sin(\lambda t) \\
&\quad + \omega_x(0)^2 \cos^2(\lambda t) + \omega_y(0)^2 \sin^2(\lambda t) - 2\omega_x(0)\omega_y(0) \cos(\lambda t) \sin(\lambda t) \\
&= \omega_x(0)^2(\sin^2(\lambda t) + \cos^2(\lambda t)) + \omega_y(0)^2(\cos^2(\lambda t) + \sin^2(\lambda t)) \\
&= \omega_x(0)^2 + \omega_y(0)^2
\end{align*}
\]

Thus

- $\omega_z$ is constant
  - rotation about axis of symmetry
- $\sqrt{\omega_x^2 + \omega_y^2}$ is constant
  - rotation perpendicular to axis of symmetry

This type of motion is often called Precession!
Circular Motion in the Body-Fixed Frame

Thus

\[
\omega(t) = \begin{bmatrix}
\omega_x(t) \\
\omega_y(t) \\
\omega_z(t)
\end{bmatrix}
= \begin{bmatrix}
cos(\lambda t) & -sin(\lambda t) & 0 \\
sin(\lambda t) & cos(\lambda t) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\omega_x(0) \\
\omega_y(0) \\
\omega_z(0)
\end{bmatrix}
= R_3(\lambda t)
\begin{bmatrix}
\omega_x(0) \\
\omega_y(0) \\
\omega_z(0)
\end{bmatrix}
\]
Circular Motion in the Body-Fixed Frame

For \( \lambda > 0 \), this is a Positive (counterclockwise) rotation, about the z-axis, of the angular velocity vector \( \omega \) as expressed in the body-fixed coordinates!
Prolate vs. Oblate

The speed of the precession is given by the natural frequency:

\[ \lambda = \frac{I_z - I_x}{I_x} \omega_z \]

with period \( T = \frac{2\pi}{\lambda} = \frac{2\pi I_x}{I_z - I_x} \omega_z^{-1} \).

**Direction of Precession:** There are two cases

**Definition 4 (Direct).**

An axisymmetric (about \( z \)-axis) rigid body is **Prolate** if \( I_z < I_x = I_y \).

**Definition 5 (Retrograde).**

An axisymmetric (about \( z \)-axis) rigid body is **Oblate** if \( I_z > I_x = I_y \).

Thus we have two cases:

- \( \lambda > 0 \) if object is **Oblate** (CCW rotation)
- \( \lambda < 0 \) if object is **Prolate** (CW rotation)

Note that these are rotations of \( \omega \), as expressed in the **Body-Fixed** Frame.
Pay Attention to the Body-Fixed Axes

The black arrow is $\vec{\omega}$.

- The body-fixed $x$ and $y$ axes are indicated with red and green dots.
- Notice the direction of rotation of $\omega$ with respect to these dots.
- The angular momentum vector is the inertial $z$ axis.
Motion in the Inertial Frame

As these videos illustrate, we are typically interested in motion in the **Inertial Frame**.

- Use of Rotation Matrices is complicated.
  - Which coordinate system to use???
- Lets consider motion relative to $\vec{h}$.
  - Which is fixed in inertial space.

We know that in Body-Fixed coordinates,

$$\vec{h} = \mathbf{I} \vec{\omega} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}$$

Now let's find the orientation of $\vec{\omega}$ and $\hat{z}$ with respect to this fixed vector.
As these videos illustrate, we are typically interested in motion in the Inertial Frame.

- Use of Rotation Matrices is complicated.
- Which coordinate system to use??
- Lets consider motion relative to $\vec{h}$.
- Which is fixed in inertial space.

We know that in Body-Fixed coordinates,

$$\vec{h} = I \vec{ω} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}$$

Now lets find the orientation of $\vec{ω}$ and $\hat{z}$ with respect to this fixed vector.

- The “Space Cone” is how $\vec{ω}$ moves in inertial coordinates
- The “Body Cone” is how $\vec{ω}$ moves with respect to the body.
Motion in the Inertial Frame

Let \( \hat{x}, \hat{y}, \) and \( \hat{z} \) define the body-fixed unit vectors.

We first note that since \( I_x = I_y \) and

\[
\vec{h} = I_x \omega_x \hat{x} + I_y \omega_y \hat{y} + I_z \omega_z \hat{z} = I_x (\omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}) + (I_z - I_x) \omega_z \hat{z} = I_x \vec{\omega} + (I_z - I_x) \omega_z \hat{z}
\]

we have that

\[
\vec{\omega} = \frac{1}{I_x} \vec{h} + \frac{I_x - I_z}{I_x \omega_z} \hat{z}
\]

which implies that \( \vec{\omega} \) lies in the \( \hat{z} - \vec{h} \) plane.
Motion in the Inertial Frame

We now focus on two constants of motion

- $\theta$ - The angle $\vec{h}$ makes with the body-fixed $\hat{z}$ axis.
- $\gamma$ - The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

Since

$$\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}$$

The angle $\theta$ is defined by

$$\tan \theta = \frac{\sqrt{h_x^2 + h_y^2}}{h_z} = \frac{I_x \sqrt{\omega_x^2 + \omega_y^2}}{I_z \omega_z} = \frac{I_x \omega_{xy}}{I_z \omega_z}$$

Since $\omega_{xy}$ and $\omega_z$ are fixed, $\theta$ is a constant of motion.
Motion in the Inertial Frame

We now focus on two constants of motion:

- $\theta$: The angle $\vec{h}$ makes with the body-fixed $\hat{z}$ axis.
- $\gamma$: The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

Since $\vec{h} = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \begin{bmatrix} I_x \omega_x \\ I_y \omega_y \\ I_z \omega_z \end{bmatrix}$, the angle $\theta$ is defined by

$$\tan \theta = \sqrt{h_x^2 + h_y^2} = \sqrt{I_x \omega_x^2 + I_y \omega_y^2} = \sqrt{I_x I_z \omega_{xy} \omega_z}$$

Since $\omega_{xy}$ and $\omega_z$ are fixed, $\theta$ is a constant of motion.

Again, $\vec{h}$ here is in the body-fixed frame.

- This is why it changes over time.
The second angle to consider is:

- \( \gamma \) - The angle \( \vec{\omega} \) makes with the body-fixed \( \hat{z} \) axis.

As before, the angle \( \gamma \) is defined by:

\[
\tan \gamma = \frac{\sqrt{\omega_x^2 + \omega_y^2}}{\omega_z} = \frac{\omega_{xy}}{\omega_z}
\]

Since \( \omega_{xy} \) and \( \omega_z \) are fixed, \( \gamma \) is a constant of motion.

- We have the relationship:

\[
\tan \theta = \frac{I_x \omega_{xy}}{I_z \omega_z} = \frac{I_x}{I_z} \tan \gamma
\]

Thus we have two cases:

1. \( I_x > I_z \) - Then \( \theta > \gamma \)
2. \( I_x < I_z \) - Then \( \theta < \gamma \) (As Illustrated)
Motion in the Inertial Frame

Figure: The case of $I_x > I_z$ ($\theta > \gamma$)

Figure: The case of $I_z > I_x$ ($\gamma > \theta$)
We illustrate the motion using the Space Cone and Body Cone

- The space cone is fixed in inertial space (doesn’t move)
- The space cone has width $|\omega - \theta|$
- The body cone is centered around the z-axis of the body.
- In body-fixed coordinates, the space cone rolls around the body cone (which is fixed)
- In inertial coordinates, the body cone rolls around the space cone (which is fixed)
Motion in the Inertial Frame

The orientation of the body in the inertial frame is defined by the sequence of Euler rotations

- \( \psi \) - \( R_3 \) rotation about \( \vec{h} \).
  - Aligns \( \hat{e}_x \) perpendicular to \( \hat{z} \).
- \( \theta \) - \( R_1 \) rotation by angle \( \theta \) about \( \vec{h}_x \).
  - Rotate \( \hat{e}_z \)-axis to body-fixed \( \hat{z} \) vector
  - We have shown that this angle is fixed!
  - \( \dot{\theta} = 0 \).
- \( \phi \) - \( R_3 \) rotation about body-fixed \( \hat{z} \) vector.
  - Aligns \( \hat{e}_x \) to \( \hat{x} \).

The Euler angles are related to the angular velocity vector as

\[
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix} = \begin{bmatrix}
\dot{\psi} \sin \theta \sin \phi \\
\dot{\psi} \sin \theta \cos \phi \\
\dot{\phi} + \dot{\psi} \cos \theta = \text{constant}
\end{bmatrix}
\]
Motion in the Inertial Frame

This comes from

\[
\mathbf{\dot{\omega}} = R_3(\phi)R_1(\theta)R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \sin \theta \sin \phi \end{bmatrix} + R_3(\phi)R_1(\theta) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + R_3(\phi) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}
\]

\[
= \begin{bmatrix} \dot{\psi} \sin \theta \sin \phi \\ \dot{\psi} \sin \theta \cos \phi \\ \dot{\psi} \cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}
\]
Motion in the Inertial Frame

To find the motion of \( \omega \), we differentiate

\[
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} =
\begin{bmatrix}
\dot{\psi} \phi \sin \theta \cos \phi \\
-\dot{\psi} \phi \sin \theta \sin \phi \\
0
\end{bmatrix}
\]

Now, substituting into the Euler equations yields

\[
\dot{\psi} = \frac{I_z}{(I_x - I_z) \cos \theta} \dot{\phi}
\]

There are two cases here:

- **Direct** precession
  - \( I_x > I_z \)
  - \( \dot{\psi} \) and \( \dot{\phi} \) aligned.

- **Retrograde** precession
  - \( I_y > I_x \)
  - \( \dot{\psi} \) and \( \dot{\phi} \) are opposite.
Recall $\dot{\omega}_x$ and $\dot{\omega}_y$ can be expressed in terms of $\omega_x$ and $\omega_y$.
Motion in the Inertial Frame

**Figure:** Retrograde Precession ($I_z > I_x$)

**Figure:** Direct Precession ($I_z < I_x$)
Mathematica Precession Demonstration
Prolate and Oblate Spinning Objects

**Figure:** Prolate Object: \( I_x = I_y = 4 \) and \( I_z = 1 \)

**Figure:** Oblate Object: Vesta
Note Bene: Precession of a spacecraft is often called nutation ($\theta$ is called the nutation angle).

- By most common definitions, for torque-free motions, $N = 0$
  - Free rotation has NO nutation.
  - This is confusing
Precession
Example: Chandler Wobble

**Problem:** The earth is 42.72 km wider than it is tall. How quickly will the rotational axis of the earth precess due to this effect?

**Solution:** for an axisymmetric ellipsoid with height \( a \) and width \( b \), we have
\[
I_x = I_y = \frac{1}{5} m (a^2 + b^2) \quad \text{and} \quad I_z = \frac{2}{5} mb^2
\]
Thus \( b = 6378 \text{km} \), \( a = 6352 \text{km} \) and we have
\( m_e = 5.974 \cdot 10^{24} \text{kg} \)
\[
I_z = 9.68 \cdot 10^{37} \text{kg m}^2, \quad I_x = I_y = 9.72 \cdot 10^{37} \text{kg m}^2
\]
If we take \( \omega_z = \frac{2\pi}{T} \cong 2\pi \text{day}^{-1} \), then we have
\[
\lambda = \frac{I_z - I_x}{I_x} \omega_z = 0.0041 \text{day}^{-1}
\]
That gives a period of \( T = \frac{2\pi}{\lambda} = 243.5 \text{days} \). This motion of the earth is known as the Chandler Wobble.

**Note:** This is only the Torque-free precession.
Precession

- Actual period is 434 days
  - Actual $I_x = I_y = 8.008 \cdot 10^{37} \, \text{kg} \cdot \text{m}^2$
  - Actual $I_z = 8.034 \cdot 10^{37} \, \text{kg} \cdot \text{m}^2$
  - Which would predict $T = \frac{2 \pi}{\sqrt{g}} = 306 \, \text{days}$

\[ \approx 8.3 \, \text{yrs} \]
\[ \approx 7.5 \, \text{rotations} \]
\[ \approx 434 \, \text{days} \]
The precession of the earth was first noticed by Euler, D’Alembert and Lagrange as slight variations in latitude.

Error partially due to fact Earth is not a rigid body (Chandler + Newcomb).

Magnitude of around 9m

Previous plot scale is milli-arc-seconds (mas)
In the next lecture we will cover

**Non-Axisymmetric rotation**
- Linearized Equations of Motion
- Stability

**Energy Dissipation**
- The effect on stability of rotation