Lecture 17: Stability of Torque-Free Motion

Case 1:

Case 2:

Damping

$\tau < 0$
In this Lecture we will cover:
Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation
Review: Euler Equations

\[ \dot{\omega}_x(t) = -\frac{I_z - I_y}{I_x} \omega_y(t)\omega_z(t) \]
\[ \dot{\omega}_y(t) = -\frac{I_x - I_z}{I_y} \omega_x(t)\omega_z(t) \]
\[ \dot{\omega}_z(t) = -\frac{I_y - I_x}{I_z} \omega_x(t)\omega_y(t) \]

**Axisymmetric Case:** \( I_x = I_y \)
- \( \dot{\omega}_z = 0 \) - \( \omega_z \) is fixed
- Equations naturally become linear.
- Allows us to solve these linear equations explicitly

**Non-axisymmetric Case:** \( I_x \neq I_y \)
- We will have to rely on linear approximation
Linearization of the Euler Equations

Linearization allows us to consider small deviations about an equilibrium.

- We need to define the equilibrium

**CASE: Stability of Spin about a principle axis.**

- Nominal motion is

\[
\omega_0(t) = \begin{bmatrix}
\omega_{x,0}(t) \\
\omega_{y,0}(t) \\
\omega_{z,0}(t)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\eta_{xy}
\end{bmatrix}
\]

- This is an equilibrium because

\[
\begin{align*}
\dot{\omega}_{x,0}(t) &= -\frac{I_z - I_y}{I_x} \omega_{y,0}(t) \omega_{z,0}(t) = 0 \\
\dot{\omega}_{y,0}(t) &= -\frac{I_x - I_z}{I_y} \omega_{x,0}(t) \omega_{z,0}(t) = 0 \\
\dot{\omega}_{z,0}(t) &= -\frac{I_y - I_x}{I_z} \omega_{x,0}(t) \omega_{y,0}(t) = 0
\end{align*}
\]
Now consider small disturbances to this equilibrium

\[ \omega(t) = \omega_0 + \Delta \omega(t) \]

Then \( \Delta \omega(t) = \omega(t) - \omega_0 \) and

\[
\Delta \dot{\omega}(t) = \dot{\omega}(t) - 0 = \begin{bmatrix}
-I_z - I_y & \omega_y(t) & \omega_z(t) \\
-I_x - I_z & \omega_x(t) & \omega_z(t) \\
-I_y - I_x & \omega_x(t) & \omega_y(t)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-I_z - I_y (\omega_{y,0} + \Delta \omega_y(t)) (\omega_{z,0} + \Delta \omega_z(t)) \\
-I_x - I_z (\omega_{x,0} + \Delta \omega_x(t)) (\omega_{z,0} + \Delta \omega_z(t)) \\
-I_y - I_x (\omega_{x,0} + \Delta \omega_x(t)) (\omega_{y,0} + \Delta \omega_y(t))
\end{bmatrix}
\]
Linearization of the Euler Equations

Now because we have assumed that $\Delta \omega$ is small, products of the form $\Delta \omega \cdot \Delta \omega$ are very small indeed. Using this observation, we make the following Approximations:

$$\Delta \omega_x \Delta \omega_y = 0, \quad \Delta \omega_x \Delta \omega_z = 0, \quad \Delta \omega_z \Delta \omega_y = 0$$

This yields the following set of linearized equations:

$$\Delta \omega(t) = \begin{bmatrix} -\frac{I_x-I_y}{I_z} \Delta \omega_y(t) (n + \Delta \omega_z(t)) \\ -\frac{I_y-I_z}{I_x} \Delta \omega_x(t) (n + \Delta \omega_z(t)) \\ -\frac{I_y-I_z}{I_x} \Delta \omega_x(t) \Delta \omega_y(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{\Delta \omega_x} \\ \dot{\Delta \omega_y} \\ \dot{\Delta \omega_z} \end{bmatrix} = \begin{bmatrix} -\frac{I_x-I_y}{I_z} n \Delta \omega_y(t) \\ -\frac{I_y-I_z}{I_x} n \Delta \omega_x(t) \\ 0 \end{bmatrix}$$

$$\Delta \omega_e = 0 \Rightarrow \Delta \omega_e = c$$
Linearization of the Euler Equations

Thus the evolution of small disturbances is governed by a set of linear equations:

\[
\begin{bmatrix}
\Delta \dot{\omega}_x(t) \\
\Delta \dot{\omega}_y(t) \\
\Delta \dot{\omega}_z(t)
\end{bmatrix}
= \begin{bmatrix}
0 & -\frac{I_z - I_y}{I_x} & 0 \\
\frac{I_x - I_z}{I_y} & 0 & 0 \\
-\frac{I_x - I_z}{I_y} & \frac{I_x - I_z}{I_y} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \omega_x(t) \\
\Delta \omega_y(t) \\
\Delta \omega_z(t)
\end{bmatrix}
\]

\(\Delta \dot{\omega}_z(t) = 0\)

- The third equation \(\Delta \dot{\omega}_z = 0\) implies \(\Delta \omega_z(0) = \text{constant}\).
- The first two equations combine to yield

\[\Delta \ddot{\omega}_x(t) = -\frac{I_z - I_y}{I_x} n \Delta \dot{\omega}_y(t) = \frac{I_z - I_y}{I_x} \frac{I_x - I_z}{I_y} n^2 \Delta \omega_x(t)\]

If we take the Laplace transform of this equation, we get

\[s^2 \hat{\Delta \omega}_x(s) = \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \hat{\Delta \omega}_x(s)\]
Stability Analysis

From
\[ s^2 \Delta \hat{\omega}_x(s) = \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \Delta \hat{\omega}_x(s) \]
we get the transfer function
\[ \hat{G}(s) = \frac{1}{s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2} = \frac{1}{\lambda(s)} \]
or if you prefer, the characteristic equation
\[ \lambda(s) = s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \]
The roots of this characteristic equation are
\[ s = \pm \sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2} \]
Consider a differential equation with characteristic equation, $\lambda(s)$:

Recall that the roots of the characteristic equation tell us about the behaviour of the variable $\Delta \omega_x$.

- The roots may be real, imaginary, or a mixture: $s = a + bi$

There are **Three Cases**:

1. **[Instability:]** If $\text{Real}(s) = a > 0$ for **any root of $\lambda(s)$**, then small disturbances will grow over time.

2. **[Stability:]** If $\text{Real}(s) = a < 0$ for **all roots of $\lambda(s)$**, then small disturbances will vanish over time.

3. **[Neutral Stability:]** If $\text{Real}(s) = a = 0$ for **any root of $\lambda(s)$**, then small disturbances will persist, but will not grow.
Stability of Torque-free Spin

\[ \lambda(s) = s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \]

Now recall the roots of \( \lambda(s) \) for the torque-free spacecraft spinning about the \( \hat{z} \) axis with angular velocity \( n \).

- The roots of \( \lambda(s) \) are

\[ s = \pm \sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y}} n \]

We can break down our stability analysis into three cases:

**CASE 1:** Spin about the major axis \( (I_z > I_x \text{ and } I_z > I_y) \).

1. In this case \( (I_z - I_y) > 0 \) and \( (I_x - I_z) < 0 \).
2. Then the roots are purely imaginary
3. \( s = \pm ib \) where \( b = \sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}} n^2 \) is real.
Stability of Torque-free Spin

\[ s = \pm \sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y}} n^2 \]

CASE 2: Spin about the minor axis \((I_z < I_x \text{ and } I_z < I_y)\).
1. In this case \((I_z - I_y) < 0\) and \((I_x - I_z) > 0\).
2. The roots are also purely imaginary.
3. \( s = \pm ib \) where \( b = \sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}} n^2 \) is real.

CASE 3: Spin about the intermediate axis \((I_y < I_z < I_x \text{ or } I_x < I_z < I_y)\).
1. In this case \((I_z - I_y)(I_x - I_z) > 0\).
2. The roots are real.
3. \( s = \pm a \) where \( a = \sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y}} n^2 > 0 \) is real.
4. One of the roots has positive real part - UNSTABLE

Thus spin about an intermediate axis is always unstable (small deviations will eventually get big!)
This effect can be visualized using Polhodes.

- Positions of the axis of rotation, $\vec{ω}$
- For fixed energy, lines are of constant angular momentum $\vec{h}$. 
Instability of the intermediate axis

**Figure:** A Deck of Cards on the ISS (Garriott)

**Figure:** Simulated Ellipsoid

**Figure:** A Textbook on the ISS (Petit)
Destabilization caused by Energy Dissipation

Summary:
- Spin about intermediate axis - **Unstable**
- Spin about major or minor axis - **Neutral Stability**

What about Disturbances?
- Fuel Sloshing
- Flexible Structures
- Heat dissipation

Problem:
- Newton’s Second Law predicts Conservation of **Momentum**
- It says nothing about **Kinetic Energy**!!!

**Question:** What is the effect of losses in Kinetic Energy?
Destabilization caused by Energy Dissipation

Summary:
- Spin about intermediate axis - Unstable
- Spin about major or minor axis - Neutral Stability

What about Disturbances?
- Fuel Sloshing
- Flexible Structures
- Heat dissipation

Problem:
- Newton’s Second Law predicts Conservation of Momentum
  - It says nothing about Kinetic Energy!!!

Question: What is the effect of losses in Kinetic Energy?

Spacecraft depicted is Explorer 1
- First American satellite
- Based on missile technology (no separation from rocket motor)
- Launched January 31, 1958
- Initially spin about minor axis.
- Quickly started precessing and decayed to spin about major axis
- Energy Dissipation from long flexible antennae
- Prompted development of Euler equations.
Destabilization caused by Energy Dissipation

**Question:** How to relate energy drain $\dot{T} < 0$ to changes in $\bar{\omega}$?

Consider the expression for Kinetic Energy:

$$2T = \omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z$$

Meanwhile, the total angular momentum is

$$h^2 = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2$$

Consider the Axisymmetric Case: $I_x = I_y$. Then

$$2T' = I_x (\omega_x^2 + \omega_y^2) + \omega_z^2 I_z$$

$$h^2 = I_x^2 (\omega_x^2 + \omega_y^2) + I_z^2 \omega_z^2$$

the second equation implies

$$\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2 \omega_z^2}{I_x^2}$$

We substitute $\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2 \omega_z^2}{I_x^2}$ into the expression for $T$ to get

$$2T = \frac{h^2 - I_z^2 \omega_z^2}{I_x} + \omega_z^2 I_z = \frac{h^2}{I_x} + \omega_z^2 I_z \left(1 - \frac{I_z}{I_x}\right)$$
Destabilization caused by Energy Dissipation

Question: How to relate energy drain $\dot{T} < 0$ to changes in $\vec{\omega}$?

Consider the expression for Kinetic Energy:

$$2T = \frac{h^2}{I_x} + \omega_z^2 I_z \left(1 - \frac{I_z}{I_x}\right)$$

- $T$ may decrease, but $h$ is invariant
- $\omega_{x,y}$ and $\omega_z$ may change as $T$ decreases.
- The expression only include $\omega_z$, however.
Destabilization caused by Energy Dissipation

Now consider the angle ($\theta$) by which $\vec{h}$ differs from $\hat{z}$.

$$\cos \theta = \frac{h_z}{h} = \frac{I_z \omega_z}{h}$$

We would like to express $T$ in terms of $\theta$.

We solve this equation for $\omega_z$ to get $\omega_z = \frac{h}{I_z} \cos \theta$.

Combining with the equation for $T$, we get

$$2T = \frac{h^2}{I_x} + \frac{h^2}{I_z} \cos^2 \theta \left(1 - \frac{I_z}{I_x}\right)$$

Taking the time-derivative, we find

$$\dot{T} = -\frac{h^2}{2I_z} 2 \cos \theta \sin \theta \left(1 - \frac{I_z}{I_x}\right) \dot{\theta}$$

$$\dot{\theta} \omega = \frac{h^2}{2I_z} 2 \cos \theta \sin \theta \left(\frac{I_z}{I_x} - 1\right) \dot{\theta}$$
Recall $\theta$ is the angle the angular momentum vector makes with the body-fixed axis.

- $\vec{h}$ and $\vec{\omega}$ are expressed in body-fixed coordinates.
Destabilization caused by Energy Dissipation

\[ \dot{T} = \frac{h^2}{I_z} \cos \theta \sin \theta \left( \frac{I_z}{I_x} - 1 \right) \dot{\theta} \]

or

\[ \dot{\theta} = \frac{I_z I_x}{h^2 \cos \theta \sin \theta \left( I_z - I_x \right)} \dot{T} \]

Recall that \( \theta \) is the angle by which \( \vec{h} \) differs from \( \hat{z} \).

- Initially, \( \theta = 0 \) - spin aligned with \( \hat{z} \) axis.

There are two cases

1. **CASE 1:** If \( I_x > I_z \), then \( \dot{T} < 0 \) implies that \( \dot{\theta} > 0 \).
   - Spin Axis \( \hat{z} \) is **UNSTABLE**

2. **CASE 2:** If \( I_x < I_z \), then \( \dot{T} < 0 \) implies that \( \dot{\theta} < 0 \).
   - Spin Axis \( \hat{z} \) is **STABLE**
The effect of energy dissipation can also be visualized using Polhodes.

- Each line has constant $\frac{h}{T}$.
- Rotation proceeds from large $T$ to small $T$. 
The effect of energy dissipation can also be visualized using Polhodes.

- Each line has constant $h^2$.
- Rotation proceeds from large $T$ to small $T$.

Polhode represents intersection of energy and inertia ellipsoids.

**Poinsot’s construction:** Take the inertia ellipsoid, hold the center a fixed distance from an inkpad and where it rolls forms one of the lines.
Theorem 1 (Major Axis Rule).

1. Spin about the **major axis is stable**
2. Spin about any other axis is unstable

Conclusion:
- Spacecraft must be fat!

Problem:
- Rockets are thin.

Solution: Dual Spinners
- Only a fat slice of the spacecraft is spun up
- Allows **nutation dampers** to stabilize the spin axis.
Dual Spinners (General Case)

A De-spun section can increase the stability about a minor axis.

\[ k_1 = \frac{I_2 - I_3}{I_3}, \quad k_3 = \frac{I_2 - I_1}{I_3}, \]

\[ \hat{\Omega}_{po} := \frac{\Omega_{po}}{\nu}, \quad \Omega_{po} = \frac{h_s}{\sqrt{I_1 I_3}} \]

- \( h_s = I_s \omega_s \) is angular momentum of spinning section
- \( \nu \) is angular speed of body about the 2-axis

Figure: Stability Regions for Dual-Spinner
Alternately, we can redefine $k_1$, $k_3$

\[ k_{1h} := k_1 + \hat{\Omega}_{po} \sqrt{\frac{1 - k_1}{1 - k_3}} \]

\[ k_{2h} := k_2 + \hat{\Omega}_{po} \sqrt{\frac{1 - k_3}{1 - k_1}} - \Xi \]

- Stable iff $k_{1h}k_{3h} > 0$. With energy dissipation: if $k_{1h} > 0$ and $k_{3h} > 0$
- $\hat{\Omega}_{po} > 0$ if body and wheel spinning in same direction.
- $\Xi$ is an energy damping term
In this Lecture we have covered:
Non-Axisymmetric rotation
  • Linearized Equations of Motion
  • Stability
Energy Dissipation
  • The effect on stability of rotation