Modern Control Systems

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Lecture 2: Mathematical Preliminaries

Mathematics - Sets and Subsets

Definition 1.

A set, D, is any collection of elements, d. Denoted $d \in D$.

Discrete Sets

- Students at ASU
- Stars in the sky
  - Sets need not be finite
- primary colors
- Natural numbers

Continuous Sets

- Space
- Arizona
- The set of all colors
- The real numbers
- Water in the ocean

Sets and Subsets

Definition 2.

A **Subset**, C, of a set, D, is a set, all of whose element are elements of D. This is denoted $C \subset D$, so that $c \in C$ implies $c \in D$

- {Rational Numbers} $\subset$ {Real Numbers}
- {MAE students at ASU} $\subset$ {students at ASU}
- {Red Dwarf Stars} $\subset$ {Stars}
- {Building ERC} $\subset$ {ASU Campus}
- {The ocean within 50ft of land} $\subset$ {The Ocean}
- {Illinois} $\subset$ {Illinois}
- {Unit Ball} $\subset$ $\{\mathbb{R}^2\}$

Notation: A subset is a set subject to constraints.

$$\{x \in D \,:\, \|x\| \leq 1\}$$

Union of Sets

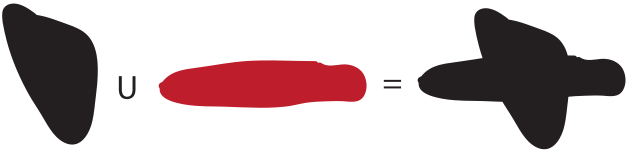
Sets can be combined to form new sets.

Definition 3.

The union of two sets, $C \cup D$ is the set whose elements are in either C or D.

$$C \cup D := \{x : x \in C \text{ or } x \in D\}$$

- $:=$ means "is defined as"



- $\{$Mexico$\} \cup \{$United States$\} \cup \{$Canada$\} = \{$North America$\}$

Intersection of Sets

Definition 4.

The intersection of 2 sets, $C \cap D$, is the set whose elements are in both C and D.

$$C \cap D := \{x \,:\, x \in C \text{ and } x \in D\}$$

- $\{\text{Mexico}\} \cap \{\text{North America}\} = \{\text{Mexico}\}$
- $\{\text{Mexico}\} \cap \{\text{United States}\} = \emptyset$
  - $\emptyset$ denotes the Null Set, the set with no elements.
- $\{x \,:\, x \geq 1\} \cap \{x \,:\, x \leq 1\} = \{1\}$
- $\{x \,:\, x < 1\} \cap \{x \,:\, x \geq 1\} = \emptyset$

Vector Spaces : Addition

Definition 5.

A set, C, has the **addition property** if for every $u, v \in C$, there exists a *unique* $w \in C$, denoted $u + v = w$, such that

1. There is a zero element, denoted 0, such that $u + 0 = u$ for any $u \in C$
2. For each $u \in C$, there is an element, denoted $-u$, such that $u + (-u) = 0$.
3. $u + (v + w) = (u + v) + x$ (Associativity)
4. $u + v = v + u$ (Commutativity)

- Natural numbers
- Matrices
- Continuous functions

Show that the unit ball does not satisfy the addition property.

Vector Spaces : Multiplication

Definition 6.

A set, C, has the **scalar multiplication property** if for every $\alpha \in \mathbb{R}$ and $u \in C$, there is a unique element, denoted $w = \alpha u \in C$, such that

- $(\alpha \cdot \beta)u = \alpha(\beta u)$ for all $\alpha, \beta \in \mathbb{R}$ and $u \in C$
- $1 \cdot u = u$.

Consider the Following Sets

- Countable numbers?
- $\mathbb{R}^n$ or $\mathbb{R}^{n \times m}$ (vectors and matrices)
- continuous functions
- The unit ball

Note that scalar multiplication is much easier than multiplication.

Vector Spaces : Definition

Definition 7.

A set, C, is a **Vector Space** if it has the addition and scalar multiplication properties and

1. $\alpha(u+v) = \alpha u + \alpha v$ for all $\alpha \in \mathbb{R}$ and $u, v \in C$. (vector distributivity)
2. $(\alpha + \beta)u = \alpha u + \beta u$ for all $\alpha, \beta \in \mathbb{R}$ and $u \in C$. (scalar distributivity)

Vector spaces are the most basic structure for which we can perform control.

- $\mathbb{R}^n$ or $\mathbb{R}^{n \times m}$ - state space
- continuous functions, $\mathcal{C}$ - control of delays or PDEs
- infinite sequences - discrete time control
- Set of polynomial functions

Vector Spaces : Cartesian Product

We can combine vector spaces using the cartesian product, $\times$

$$C \times D := \{(c, d) \;:\; c \in C \text{ and } d \in D\}$$

Lemma 8.

If C and D are vector spaces, $C \times D$ is a vector space

Proof.

- Addition Property

$$(c_1, d_1) + (c_2, d_2) = (c_1 + c_2, d_1 + d_2)$$

- Scalar multiplication Property

$$\alpha(c_1, d_1) = (\alpha c_1, \alpha d_1)$$

□

Vector Spaces : Cartesian Product

Examples:

- Vectors and matrices

$$\mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \; : \; x_i \in \mathbb{R} \right\}$$

- products of vectors and functions

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; : \; x_1 \in \mathbb{R}, \quad x_2 \in \mathcal{C} \right\}$$

Definition 9.

A **subspace** is a subset of a vector space which is also a vector space using the same definitions of addition and multiplication.

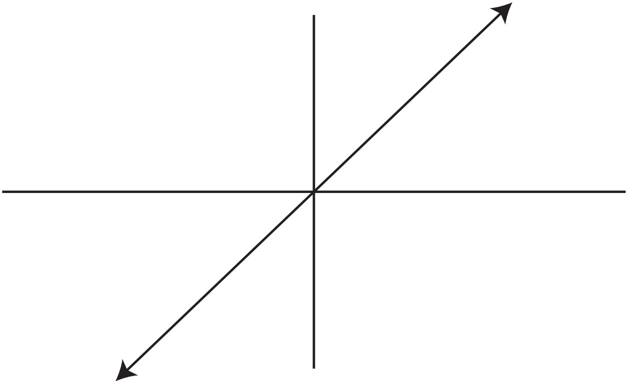


Figure: A subspace of $\mathbb{R}^2$

Vector Spaces : Subspaces

Any element, v of a vector space, C can define a subspace as

$$S_v := \{s \in C \;:\; s = \alpha v,\, \alpha \in \mathbb{R}\}$$

Proof.

- Addition: $s_1 + s_2 = \alpha_1 v + \alpha_2 v = (\alpha_1 + \alpha_2)v$
- Multiplication: $\beta s = \beta\alpha v = (\alpha\beta)v$

$\square$

- A 1-dimensional subspace
- The line which passes through 0 and v.

Note that a subspace must contain the origin in order to be a vector space.

Vector Spaces : Subspaces

A set of points, $\{v_1, v_2, \cdots, v_n\} \subset C$ defines a minimum subspace as

$$S_{\{v_i\}} := \{s \in C \;:\; x = \sum_i \alpha_i v_i,\; \alpha_i \in \mathbb{R}\}$$

If the v_i are independent ($v_j \neq \sum_{i \neq j} \alpha_i v_i$), then the subspace is n-dimensional

Examples of Subspaces:

- lines and planes through the origin (subspaces of $\mathbb{R}^3$)
- polynomials (subspace of continuous functions)
- The space itself
- bounded functions (subspace of functions)
- $\mathbb{R}^{m-1}$ in $\mathbb{R}^m$

$$\{x \in \mathbb{R}^m \;:\; x_m = 0\}$$

Vector Spaces : Other Important Subspaces

Matrix Transpose If $A \in \mathbb{R}^{m \times n}$, the transpose is $A^T \in \mathbb{R}^{n \times m}$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}, \quad \text{and} \quad A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \ldots & a_{mn} \end{bmatrix}$$

- For complex matrices $A \in \mathbb{C}^{m \times n}$, a_{ij}^* is the complex conjugate of a_{ij} and

$$A^* = \begin{bmatrix} a_{11}^* & \cdots & a_{m1}^* \\ \vdots & \ddots & \vdots \\ a_{1n}^* & \ldots & a_{mn}^* \end{bmatrix}$$

Definition 10.

$A \in \mathbb{R}^{m \times n}$ is **self-adjoint** if $A = A^*$ or $A = A^T$.

- The set of self-adjoint matrices is a subspace of squares matrices, $\mathbb{R}^{n \times n}$

$$\mathbb{S}^n := \{A \in \mathbb{R}^{n \times n} \; : \; A = A^T\}$$

- For complex matrices, denoted $\mathbb{H}^n := \{A \in \mathbb{C}^{n \times n} \; : \; A = A^*\}$.

Vector Spaces : Other Important Subspaces

Consider polynomials. e.g.

$$p(x) = 3xy + y^2 + z^2 y + 4$$

Polynomials can be written in a standard form

$$p(x) = \sum_i a_i x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$$

Where

- $x_1, x_2, \ldots, x_n$ are the variables (instead of x,y,z)
- a_i are the coefficients, (e.g. $a_1 = 3$, $a_2 = 1$, $a_3 = 1$, $a_4 = 4$)
- $\alpha_{i,1}, \ldots, \alpha_{i,n}$ are the powers of the ith term. e.g.

$$\alpha_1 = (1,1,0), \quad \alpha_2 = (0,2,0), \quad \alpha_3 = (0,1,2), \quad \alpha_4 = (0,0,0)$$

- Each term $x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}}$ is called a monomial. e.g. xy or $z^2 y$
  - Each monomial has degree given by the sum of its terms

$$\text{degree}(z^2 y) = \sum_j \alpha_{3,j} = 0 + 1 + 2 = 3$$

Vector Spaces : Other Important Subspaces

Definition 11.

A polynomial is **homogeneous** is all monomials are of the same degree. i.e. there exists a $c \in \mathbb{N}$ such that

$$\sum_j \alpha_{i,j} = c \qquad \text{for all } i$$

We have the following *subspaces*

- The space of homogeneous polynomials

$$H := \{(a, \alpha) \in \mathbb{R}^K \times \mathbb{R}^{K,n} \;:\; \sum_j \alpha_{i,j} = \sum_j \alpha_{k,j} \text{ for all } i, k = 1, \ldots, K\}$$

- The space of homogeneous polynomials of degree d

$$H_d := \{(a, \alpha) \in \mathbb{R}^K \times \mathbb{R}^{K,n} \;:\; \sum_j \alpha_{i,j} = d \text{ for all } i = 1, \ldots, K\}$$

Vector Spaces : Review

Which of the following are subspaces?

- The hypercube: $\{x \; : \; |x_i| \leq 1\}$.
- The union of two subspaces.
- The intersection of two subspaces.
- Rational numbers in the real numbers.
- The set of integrable functions.
- The line through $(0, 1)$ and $(1, 0)$.

Basis and Dimension

Suppose $v_1, \ldots, v_n$ are elements of a vector space, X.

Definition 12.

The **Span** of $v_1, \ldots, v_n$, denoted $\text{span}\{v_1, \ldots, v_n\}$, is the smallest subspace which contains $v_1, \ldots, v_n$.

$$\text{span}\{v_1, \ldots, v_n\} := \{\sum_{i=1}^{n} \alpha_i v_i \ : \ \alpha_i \in \mathbb{R}\}$$

Examples: The canonical basis for $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\text{span}\{e_1, e_2, e_3\} = \mathbb{R}^3, \qquad \text{span}\{e_1, e_2, \ldots, e_n\} = \mathbb{R}^n$$

Basis and Dimension

Other Examples

- Fourier basis functions

$$e_1 = \sin x, \qquad e_2 = \sin 2x$$

- Monomials

$$e_0 = 1, \quad e_1 = x, \quad e_2 = x^2, \quad e_3 = x^3$$

$$\text{span}\{e_0, e_1, e_2, e_3\} = \text{polynomials in } x \text{ of degree 3 or less}$$

$$H_0[x,y] = \text{span}\{1\}, \quad H_1[x,y] = \text{span}\{x, y\},$$
$$H_2[x,y] = \text{span}\{x^2, xy, y^2\}, \quad H_3[x,y] = \text{span}\{x^3, x^2y, xy^2, y^3\}$$

Basis and Dimension

We can now define the dimension of a vector space

Definition 13.

A **basis** for vector space X is a collection of elements, $x_i \in X$ such that

$$\text{span}\{x_i\} = X$$

Definition 14.

The dimension of X is the minimum number of elements needed to form a basis for X.

- X is **finite-dimensional** if the dimension of X is finite.
- X is **infinite-dimensional** if it admits no finite basis.
- A **minimal basis** is a basis $v_1, \ldots, v_n$ which for any $x \in X$,

  $$x = \sum_{i=1}^{n} a_i v_i$$

  has a unique solution a.