Modern Control Systems

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Lecture 5: Controllability and Observability

State-Space

The standard state-space form is

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

State-space reflects an approach based on **internal dynamics** as opposed to input-output maps.

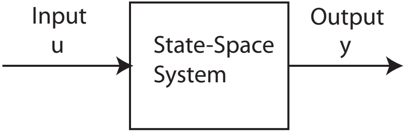
- For a given mapping, $G : u \mapsto y$, the choice of A, B, C, D is not unique.

Solving the Equations

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



Basic Question: Given an input function, $u(t)$, what is the output?
Solution: Solve the differential Equation.

Example: The equation

$$\dot{x}(t) = ax(t), \qquad x(0) = x_0$$

has solution

$$x(t) = e^{at}x_0,$$

But we are interested in Matrices. Can we define the matrix exponential?

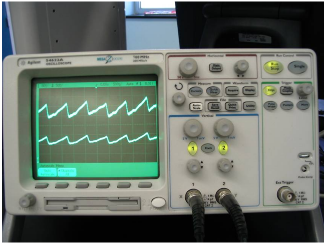
The Solution to State-Space

Ignore Inputs and Outputs:

The equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

has solution $$x(t) = e^{At}x_0$$



The function e^{At} must satisfy the following

$$e^0 = I, \qquad \text{and} \qquad \frac{d}{dt}e^{At} = Ae^{At}$$

For scalars, the matrix exponential is defined as

$$e^a = 1 + a + a^2/2 + \cdots + \frac{a^n}{n!} + \cdots$$

We define the exponential for matrices is defined the same way as scalars

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

The Solution to State-Space

The matrix exponential has the following properties

- $e^0 = I$

$$e^0 = I + 0 + \frac{1}{2}0^2 + \frac{1}{6}0^3 + \cdots = I$$

- $e^{M^*} = \left(e^M\right)^*$

$$\begin{aligned} e^{M^*} &= I + M^* + \frac{1}{2}(M^*)^2 + \frac{1}{6}(M^*)^3 + \cdots + \frac{1}{k!}(M^*)^k + \cdots \\ &= I + M^* + (\frac{1}{2}M^2)^* + \left(\frac{1}{6}M^3\right)^* + \cdots + \left(\frac{1}{k!}M^k\right)^* + \cdots \\ &= \left(I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \cdots + \frac{1}{k!}M^k + \cdots\right)^* \end{aligned}$$

The Solution to State-Space

- $\frac{d}{dt}e^{At} = Ae^{At}$

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{k!}(At)^k + \cdots\right)\\ &= 0 + A + \frac{2}{2}A(At) + \frac{3}{6}A(At)^2 + \cdots + \frac{k}{k!}A(At)^k + \cdots\\ &= A\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{(k-1)!}(At)^{k-1} + \cdots\right)\end{aligned}$$

However,

$$e^{M+N} \neq e^M e^N$$

Unless, $MN = NM$.

Find the output given the input

The equation
$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$
has solution
$$x(t) = e^{At}x_0$$

Proof.

Let $x(t) = e^{At}x_0$, then

- $\dot{x}(t) = Ae^{At}x_0 = Ax(t)$.
- $x(t) = e^{0}x_0 = x_0$

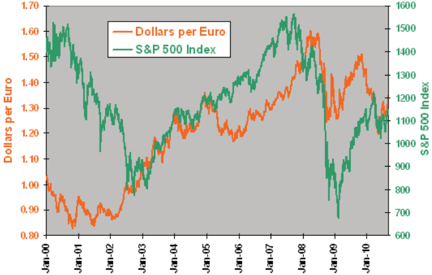
□

What happens when we add an input instead of an initial condition?

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$



The equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0$$

has solution

$$x(t) = \int_0^t e^{A(t-s)}Bu(s)ds$$

Proof.

Check the solution:

$$\dot{x}(t) = e^0 Bu(t) + A\int_0^t e^{A(t-s)}Bu(s)ds$$
$$= Bu(t) + Ax(t)$$

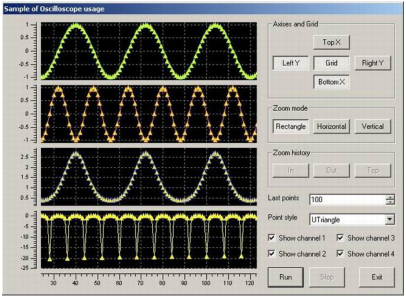
□

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

Now that we have $x(t)$, finding $y(t)$ is easy

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \end{aligned}$$



Conclusion: Given $u(t)$, only one integration is needed to find $y(t)$!
Note that the state, x, doesn't appear!!

We have solved the problem.

Calculating the Output

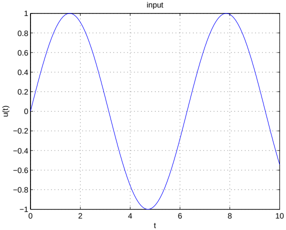
Numerical Example, $u(t) = \sin(t)$

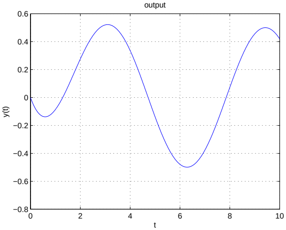
State-Space:

$$\dot{x} = -x(t) + u(t)$$
$$y(t) = x(t) - .5u(t) \qquad x(0) = 0$$
$$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$$

Solution:

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t) \\ &= e^{-t}\int_0^t e^s \sin(s)ds - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^s(\sin s - \cos s)|_0^t\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}e^{-t}\left(e^t(\sin t - \cos t) + 1\right) - \frac{1}{2}\sin(t) \\ &= \frac{1}{2}\left(e^{-t} - \cos t\right) \end{aligned}$$





Stability

$$\dot{x}(t) = Ax(t)$$

There are several notions of stability.

- All notions are equivalent for linear systems.

Definition 1.

A differential equation is stable if any solution $x(t)$ satisfies

$$\lim_{t \to \infty} x(t) = 0$$

Stability

The unique solution has the form $x(t) = e^{At}x_0$.

$$\dot{x}(t) = Ax(t)$$

Question: Is it stable?
Suppose A is *diagonalizable*, so $A = T\Lambda T^{-1}$, so that

$$A^k = T\Lambda T^{-1}T\Lambda T^{-1} \cdots T\Lambda T^{-1} = T\Lambda^k T^{-1}$$

We conclude that

$$\begin{aligned} e^{At} &= \left(TT^{-1} + (T\Lambda T^{-1})t + \frac{1}{2}(T\Lambda^2 T^{-1})t^2 + \cdots + \frac{1}{k!}(T\Lambda^k T^{-1})t^k + \cdots \right) \\ &= T\left(I + (\Lambda t) + \frac{1}{2}(\Lambda t)^2 + \cdots + \frac{1}{k!}(\Lambda t)^k + \cdots \right) T^{-1} \\ &= Te^{\Lambda t}T^{-1} \end{aligned}$$

Stability

But we can see that $e^{\Lambda t}$ converges

$$e^{\Lambda t} = I + \Lambda t + \cdots + \frac{1}{k!}\Lambda^k t^k + \cdots$$
$$= \begin{bmatrix} 1 + \lambda_1 t + \cdots & & \\ & \ddots & \\ & & 1 + \lambda_n t + \cdots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

The solution is a linear combination of the functions of the form $e^{\lambda_i t}$.

- $\lim_{t\to\infty} e^{\lambda_i t} \to 0$ if and only if $\mathrm{Re}\,\lambda_i(A) < 0$ for all i.

Stability

Inconveniently, not all matrices are diagonalizable.

- However, all matrices are Jordan diagonalizable.
  - $A^k = TJ^kT^{-1}$, where J
- Hence $e^{At} = Te^{Jt}T^{-1}$

Consider a single Jordan block $J_i = \lambda_i I + N$.

- Convenient because $\lambda_i I$ and N commute.
  - Hence $e^{\lambda_i I + N} = e^{\lambda_i I}e^N$.

$$e^{J_i t} = e^{\lambda_i t + Nt} = e^{\lambda_i t}e^{Nt}$$

Conveniently $N^d = 0$, so the series expansion terminates

$$e^{Nt} = 1 + N + \frac{1}{2}N^2 + \cdots + \frac{1}{(k-1)!}N^{k-1}t^{k-1}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & & \\ & \ddots & \\ & & e^{\lambda_i t} \end{bmatrix} \left[1 + N + \frac{1}{2}N^2 + \cdots + \frac{1}{(k-1)!}N^{k-1}t^{k-1}\right]$$

This time all terms have the form $t^i e^{\lambda t}$ for $i \leq k$

- $\lim_{t\to\infty} t^i e^{\lambda t} = 0$ if and only if $\operatorname{Re}\lambda < 0$

Stability

Now consider the general case $A = TJT^{-1}$ where

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix}$$

Then

$$e^{At} = Te^{Jt}T^{-1} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_n t} \end{bmatrix} T^{-1}$$

e^{At} is entirely composed of terms of the form

$$\frac{e^{\lambda_i t} t^k}{k!}$$

We conclude that $\dot{x}(t) = Ax(t)$ is stable if and only if $\operatorname{Re} \lambda < 0$.

Stability

Definition 2.

A is **Hurwitz** if $\operatorname{Re}\lambda_i(A) < 0$ for all i.

Theorem 3.

$\dot{x}(t) = Ax(t)$ is stable if and only if A is Hurwitz.