Modern Control Systems

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Lecture 6: Controllability and Observability

Controllability

First add an input $u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = 0$$

The solution is

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

Use Leibniz rule for differentiation of integrals

$$\begin{aligned}\dot{x}(t) &= e^{A(t-t)} Bu(t) + \int_0^t A e^{A(t-s)} Bu(s) ds \\ &= Bu(t) + Ax(t)\end{aligned}$$

Controllability asks whether we can "control" the system states through appropriate choice of $u(t)$.

- Note that we do not care how $u(t)$ is chosen.

We start with a weaker definition

Controllability

Definition 1.

For a given (A, B), the **state x_f is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} Bu(s) ds$$

Definition 2.

The **system (A, B) is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

For a fixed t, the set of reachable states is defined as

$$R_t := \{x \;:\; x = \int_0^t e^{A(t-s)} Bu(s) ds \text{ for some function } u.\}$$

Controllability

The mapping $\Gamma_t : u \mapsto x_f$ is linear. Let $u = \alpha u_1 + \beta u_2$

$$\begin{aligned}\Gamma_t u &= \int_0^{T_f} e^{A(T_f - s)} B\left(\alpha u_1(s) + \beta u_2(s)\right) ds \\ &= \alpha \int_0^{T_f} e^{A(T_f-s)} B u_1(s) ds + \beta \int_0^{T_f} e^{A(T_f-s)} B u_2(s) ds \\ &= \alpha \Gamma_t u_1 + \beta \Gamma_t u_2\end{aligned}$$

Thus $R_t = \text{Image}(\Gamma_t)$.

- R_t is a subspace.

Definition 3.

For a given system (A, B), the **Controllability Matrix** is

$$C(A,B) := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Controllability

In Williams-Lawrence, the controllability matrix is denoted P.

Definition 4.

For a given (A, B), the **Controllable Subspace** is

$$C_{AB} = \text{Image}\begin{bmatrix}B & AB & A^2B & \cdots & A^{n-1}B\end{bmatrix}$$

Definition 5.

The system (A, B) is **controllable** if

$$C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$$

Question: How does R_t relate to C_{AB}?

Controllability

Definition 6.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W_t := \int_0^t e^{As} B B^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 7.

For any $t \geq 0$,

$$R_t = C_{AB} = \textit{Image}\ (W_t)$$

or

$$\text{Image } \Gamma_t = \text{Image } C(A, B) = \text{Image } (W_t)$$

Controllability

The most important consequence is

- R_t does not depend on time!

If you can get there, you can get there arbitrarily fast.
This says nothing about how you get $u(t)$

- This $u(t)$ comes from the proof (and W_t)

We can test reachability of a point x by testing

$$x \in \text{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

The system is controllable if $W_t > 0$. Summary

1. R_t is the set of reachable points
2. $C(A, B)$ is a fixed matrix, easily computable.
3. We need to find $u(t)$

Controllability

The following is a seminal result in state-space theory.

Theorem 8 (Cayley-Hamilton Theorem).

If

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

then

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_0 I = 0$$

Proof Sketch.

The same principle as deriving the solution. Denote

$$\text{char}_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \det(sI - A)$$

Then if $A = T\Lambda T^{-1}$

$$\text{char}_A(A) = T\text{char}_A(\Lambda)T^{-1} = T \begin{bmatrix} \text{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1}$$

Controllability

Sketch.

But the λ_i are eigenvalues of A, so

$$\text{char}_A(\lambda) = \det(\lambda I - A) = 0$$

hence

$$\text{char}_A(A) = T \begin{bmatrix} \text{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1} = T \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} T = 0$$

The same approach works for Jordan Blocks. □

Cayley-Hamilton says

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

thus $A^n \in \text{span}(A^{n-1}, \cdots, I)$.

Controllability

Proof.

We need to show that $\text{Im}(W_t) = \text{Im}(C(A,B)) = R_t$. To do this, we will prove that

- $\text{Im}(W_t) \subset R_t$
- $R_t \subset \text{Im}(C(A,B))$
- $\text{Im}(C(A,B)) \subset \text{Im}(W_t)$

We begin by showing that $R_t \subset \text{image} C_{AB}$ for any $t \geq 0$. Expand

$$e^{At} = \left[I + At + \cdots + \frac{A^m t^m}{m!} + \cdots \right]$$

By Cayley-Hamilton

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

Grouping by A^i, we get

$$e^{At} = \left[I\phi_0(t) + A^1\phi_1(t) + \cdots + A^{n-1}\phi_{n-1}(t) \right]$$

for some scalar functions $\phi_i(t)$.

Controllability

Proof.

Because the ϕ_i are scalars,

$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)} B u(s) ds \\ &= B \int_0^t \phi_0(t-s) u(s) ds + \cdots + A^{n-1} B \int_0^t \phi_{n-1}(t-s) u(s) ds\end{aligned}$$

Let

$$y_i = \int_0^t \phi_i(t-s) u(s) ds,$$

then

$$\begin{aligned}\Gamma_t u &= B y_0 + \cdots + A^{n-1} B y_{n-1} \\ &= \begin{bmatrix} B & \cdots & A^{n-1} B \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = C(A,B) \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}\end{aligned}$$

Thus $\Gamma_t u \in \text{image} \begin{bmatrix} B & \cdots & A^{n-1} B \end{bmatrix}$. Therefore, $R_t \subset C_{AB}$. □

Controllability

We need to prove

- $\mathrm{Im}(W_t) \subset R_t$
- $R_t \subset \mathrm{Im}(C(A,B))$
- $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$

So far, we have show that

$$R_t \subset C_{AB}$$

Next, we will show that

$$\mathrm{Im}(W_t) \subset R_t$$

Controllability: $\text{Im}(W_t) \subset R_t$

Proposition 1.

$$\text{Im}(W_t) \subset R_t$$

Proof.

First, suppose that $x \in \text{Im}(W_t)$ for some $t > 0$. Then $x = W_t z$ for some z.

- Now let $u(s) = B^T e^{A^T(t-s)} z$. Then

$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)} B u(s) ds \\ &= \int_0^t e^{A(t-s)} B B^T e^{A^T(t-s)} z ds \\ &= W_t z = x\end{aligned}$$

- Thus $x \in \text{Im}(\Gamma_t) = R_t$.

We conclude that $\text{Im}(W_t) \subset R_t$ □

Proof by Contradiction: Im$(C(A,B)) \subset$ Im(W_t)

The last proof is a proof by contradiction.

> *Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth.*

There are two mutually exclusive possibilities

1. $x \in$ Im$(C(A,B))$ implies $x \in$ Im(W_t)
2. There exists some $x \in$ Im$(C(A,B))$ such that $x \notin$ Im(W_t) .

We eliminate the second possibility by showing that:

- If $x \notin$ Im(W_t) then $x \notin$ Im$(C(A,B))$.

In shorthand:

$$(\neg 2 \Rightarrow \neg 1) \;\Leftrightarrow\; (1 \rightarrow 2)$$

Also, recall:

Theorem 9.

For any $M \in \mathbb{R}^{n\times m}$, $[\text{Im}(M)]^{\perp} = \text{Ker}\left[M^T\right]$.

Proof by Contradiction: $\text{Im}(C(A, B)) \subset \text{Im}(W_t)$

Theorem 10.

$$\textit{Im}(C(A, B)) \subset \textit{Im}(W_t).$$

Proof.

Suppose $x \notin \text{Im}(W_t)$. Then $x \in \text{Im}(W_t)^\perp$.

- As we have shown, this means $x \in \ker(W_t)$, so $W_t x = 0$.
- Thus

$$\begin{aligned} x^T W_t x &= \int_0^t x^T e^{A(t-s)} B B^T e^{A^T(t-s)} x ds \\ &= \int_0^t u(s)^T u(s) ds = 0 \end{aligned}$$

  where $u(s) = B^T e^{A^T(t-s)} x$.
- This implies $u(s) = B^T e^{A^T(t-s)} x = 0$ for all $s \in [0, t]$.

Proof by Contradiction: $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$

Proof.

- This means that for all $s \in [0,t]$,

$$\frac{d^k}{ds^k} B^T e^{A^T s} x = B^T \left(A^T\right)^k e^{A^T s} x = 0$$

- At $s = 0$, this implies $B^T \left(A^T\right)^k x = 0$ for all k.
- We conclude that

$$x^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$$

- Thus $C(A,B)^T x = 0$, so $x \in \ker C(A,B)^T$. As before, this means $x \in \mathrm{Im}(C(A,B))^\perp$.
- We conclude that $x \notin \mathrm{Im}(C(A,B))$. This proves by contradiction that $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$.

□

Summary: $R_t = \text{Im}(W_t) = \text{Im}(C(A,B))$

We have shown that

$$R_t = \text{Im}(W_t) = \text{Im}(C(A,B))$$

Moreover, we have shown that for any $x_d \in R_{T_f}$, we can find a controller

- Choose any z such that $x_d = W_{T_f} z$. ($z = W_{T_f}^{-1} x$ if W_{T_f} is invertible)
- Let $u(t) = B^T e^{A^T (T_f - t)} z$.
- Then the system $\dot{x}(t) = Ax(t) + B(t)$ with $x(0) = 0$ has solution with $x(T_f) = x_d$.
- $x_d = \Gamma_{T_f} u$.