Modern Control Systems

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Lecture 7: Canonical Forms and Stabilizability

Representation and Controllability

Question: Is the representation (A, B, C, D) of the system $y = Gu$,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

unique?

Question: Do there exist $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ such that y and u also satisfy,

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t)$$
$$y(t) = \hat{C}x(t) + \hat{D}u(t)$$

Answer: Of Course! Recall the similarity transform: $z(t) = Tx(t)$ for any invertible T. Then y and u also satisfy,

$$\begin{aligned}\dot{z}(t) &= T\dot{x}(t) = TAx(t) + TBu(t) \\ &= TAT^{-1}z(t) + TBu(t) \\ y(t) &= Cx(t) + Du(t) \\ &= CT^{-1}z(t) + Du(t)\end{aligned}$$

Representation and Controllability

Thus the pair $(TAT^{-1}, TB, CT^{-1}, D)$ is also a representation of the map $y = Gu$.

- Furthermore $x(t) \to 0$ if and only if $z(t) \to 0$.
- So internal stability is unaffected.

Controllability is Unaffected:

$$\begin{aligned}
&C(TAT^{-1}, TB)\\
&= \begin{bmatrix} TB & TAT^{-1}TB & TAT^{-1}TAT^{-1}TB & \cdots & TA^{n-1}B \end{bmatrix}\\
&= TC(A, B)
\end{aligned}$$

Invariant Subspaces

Definition 1.

A subspace, $W \subset X$, is **Invariant** under the operator $A : X \to X$ if $x \in W$ implies $Ax \in W$.

For a linear operator, only subspaces can be invariant.

Proposition 1.

If W if A-invariant, then there exists an invertible T, such that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad \textit{and} \qquad TW = \textit{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

That is, for any $x \in W$, $Tx = \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix}$, which is clearly $\bar{A}$-invariant.

Invariant Subspaces

Proposition 2.

C_{AB} *is* A*-invariant.*

Proof.

The proof is direct. If $x \in C_{AB}$, there exists a z such that $x = C(A, B)z$. Now examine $Ax = AC(A, B)z$.

$$A \cdot C(A, B) = A \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix}$$

But, by Cayley-Hamilton,

$$A^n = \sum_{i=0}^{n-1} a_i A^i$$

so we can write

$$\begin{aligned} Ax = AC(A, B)z &= \begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix} z \\ &= \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} z_n a_0 \\ z_1 + z_n a_1 \\ \vdots \\ z_{n-1} + z_n a_{n-1} \end{bmatrix} \in C_{AB} \end{aligned}$$

Controllability Form

Since C_{AB} is an invariant subspace of A, there exists an invertible T such that

$$TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$$

and $Tx = \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}$ for any $x \in C_{AB}$.

- Clearly $B \in C_{AB}$.
- Thus $TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$.

Definition 2.

The pair (A, B) is in **Controllability Form** when

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

and the pair (A_{11}, B_1) is controllable.

Controllability Form

When a system is in controllability form, the dynamics have special structure

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$$
$$\dot{x}_2(t) = A_{22}x_2(t)$$

The x_2 dynamics are autonomous.

- Cannot be stabilized or controlled.

We can formulate a procedure for putting a system in Controllability Form

1. Find an orthonormal basis, $\begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}$ for C_{AB}.
    - Gramm-Schmidt on columns of $C(A, B)$
2. Complete the basis in $\mathbb{R}^n$: $\begin{bmatrix} v_{r+1} & \cdots & v_n \end{bmatrix}$.
3. Define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$.
4. Construct $\bar{A} = TAT^{-1}$ and $\bar{B} = TB$
    - Works for ANY invariant subspace.

Controllability Form

Example

Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Construct $C(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$A^2B = A(AB) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$C(A,B) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\operatorname{rank} C(A,B) = 2 < n = 3$ which means not controllable.

Controllability Form

Example Continued

Using Gramm-Schmidt, we can construct an orthonormal basis for C_{AB}

$$C_{AB} = \text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} = \text{span}\left\{v_1, v_2\right\}$$

Let $v_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. Then

$$T^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I$$

So $T = I$, which is because the system is already in controllability form. We could also have used

$$T^{-1} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \quad \text{to get} \quad TAT^{-1} = \begin{bmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & -1\end{bmatrix} = A$$

Stabilizability

Stabilizability is weaker than controllability

Definition 3.

The pair (A, B) is stabilizable if for any $x(0) = x_0$, there exists a $u(t)$ such that $x(t) = \Gamma_t u$ satisfies

$$\lim_{t\to\infty} x(t) = 0$$

- Again, no restriction on $u(t)$.
- Weaker than controllability
  - **Controllability:** Can we drive the system to $x(T_f) = 0$?
  - **Stabilizability:** Only need to *Approach* $x = 0$.
- Stabilizable if uncontrollable subspace is naturally stable.

Stabilizability

Consider the system in Controllability Form.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

Note that

$$\dot{x}_2(t) = A_{22} x_2(t)$$

and so, we can solve explicitly

$$x_2(t) = e^{A_{22}t} x_2(0)$$

Clearly A_{22} must be Hurwitz if (A, B) is stabilizable.

- Necessary and Sufficient

PBH Test

Lemma 4.

The pair (A, B) is stabilizable if and only if A_{22} is Hurwitz.

This is an test for stabilizability, but requires conversion to controllability form.

- A more direct test is the PBH test

Theorem 5 (PBH Test).

The pair (A, B) is

- **Stabilizable** *if and only if* $\text{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}^+$
- **Controllable** *if and only if* $\text{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ *for all* $\lambda \in \mathbb{C}$

Note: We need only check the eigenvalues λ

- Condition implies $x^T B \neq 0$ for any left eigenvector of A.
- There is also a PBH test for observability/detectability (Coming Soon)

PBH Test

Proof: Controllable if and only if $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

Proof.

We will use proof by contrapositive. $(\neg 2 \Rightarrow \neg 1)$. Suppose $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} < n$.

- Thus $\dim\left(\text{Im}\begin{bmatrix}\lambda I - A & B\end{bmatrix}\right) < n$
- There exists an x such that $x^T\begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$.
- Thus $\lambda x^T = x^T A$ and $x^T B = 0$
- Thus $x^T A^2 = \lambda x^T A = \lambda^2 x^T$.
- Likewise $x^T A^k = \lambda^k x^T$.
- Thus

$$x^T C(A,B) = x^T\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = x^T\begin{bmatrix}B & \lambda B & \cdots & \lambda^{n-1}B\end{bmatrix}$$
$$= \begin{bmatrix}0 & \cdots & 0\end{bmatrix}$$

- Thus $\dim[\text{Im} C(A,B)] < n$, which means *Not Controllable*. $(\neg 2 \Rightarrow \neg 1)$.
- We conclude that controllable implies $\operatorname{rank}\begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$.

PBH Test

Proof.

For the second part, we will also use proof by contrapositive. $(\neg 1 \Rightarrow \neg 2)$.
Suppose (A,B) is not controllable. Then there exists an invertible T such that

$$TAT^{-1} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \qquad TB = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$$

Now let λ be an eigenvalue of $\hat{A}_{22}^T$ with eigenvector $\hat{x}$. $\hat{A}_{22}^T \hat{x} = \lambda \hat{x}$. Thus $\hat{x}^T \hat{A}_{22} = \lambda \hat{x}^T$. Now define x as

$$x = T^T \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}, \qquad \text{then} \quad x^T = \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T$$

Then

$$\begin{aligned} x^T \begin{bmatrix} \lambda I - A & B \end{bmatrix} &= x^T T^{-1} \begin{bmatrix} \lambda T - TAT^{-1}T & TB \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T TT^{-1} \begin{bmatrix} \lambda T - \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} T & \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T T - \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} T & \begin{bmatrix} 0 \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \end{bmatrix} \end{aligned}$$

PBH Test

Proof.

$$
\begin{aligned}
x^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} &= \left[\lambda \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T T - \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T \begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ 0 & \hat{A}_{22}\end{bmatrix} T \quad \begin{bmatrix}0\\ \hat{x}\end{bmatrix}^T \begin{bmatrix}\hat{B}_1\\ 0\end{bmatrix}\right]\\
&= \begin{bmatrix}\begin{bmatrix}0 & \lambda \hat{x}^T\end{bmatrix} T - \begin{bmatrix}0 & \hat{x}^T \hat{A}_{22}\end{bmatrix} T & 0\end{bmatrix}\\
&= \begin{bmatrix}0 & \hat{x}^T \begin{bmatrix}\lambda I - \hat{A}_{22}\end{bmatrix} & 0\end{bmatrix} T = 0
\end{aligned}
$$

- Thus $x^T \begin{bmatrix}\lambda I - A & B\end{bmatrix} = 0$.
- Thus $\text{rank} \begin{bmatrix}\lambda I - A & B\end{bmatrix} < n$.
- Finally $(\neg 1 \Rightarrow \neg 2)$.
- We conclude that $\text{rank} \begin{bmatrix}\lambda I - A & B\end{bmatrix} = n$ implies controllability.

□

Single Input Controllability

Definition 6.

A **Companion Matrix** is any matrix of the form:

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \cdots & & -a_{n-1} \end{bmatrix}$$

A companion matrix has the convenient property that

$$\det(sI - A) = \sum_{i=0}^{n-1} a_i s^i = a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n$$

Single Input Controllability

Theorem 7.

Suppose (A, B) *is controllable.* $B \in \mathbb{R}^{n\times 1}$*. Then there exists an invertible* T *such that*

$$TAT^{-1} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & & \\ & & 0 & 1 \\ -a_0 & & & -a_{n-1} \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This is **Controllable Canonical Form**

- Different from controllability form
- This is useful for reading off transfer functions

$$G(s) = C(sI - A)^{-1}B + D$$

which has a denominator

$$\det(sI - A) = a_0 + \cdots + a_{n-1}s^{n-1}$$