Modern Control Systems

Matthew M. Peet
Arizona State University

Lecture 8: Eigenvalue Assignment

Eigenvalue Assignment

Static Full-State Feedback

The problem of designing a controller

- We have touched on this problem in reachability
  - $u(t) = B^T e^{A(T_f - t)} T^{-1} z_f$
  - This controller is open-loop
- It assumes perfect knowledge of system and state.

Problems

- Prone to Errors, Disturbances, Errors in the Model

Solution

- Use continuous measurements of state to generate control

Static Full-State Feedback Assumes:

- We can directly and continuously measure the state $x(t)$
- Controller is a static linear function of the measurement

$$u(t) = Fx(t), \qquad F \in \mathbb{R}^{m \times n}$$

Eigenvalue Assignment

Static Full-State Feedback

State Equations: $u(t) = Fx(t)$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ &= Ax(t) + BFx(t)\\ &= (A + BF)x(t)\end{aligned}$$

Stabilization: Find a matrix $F \in \mathbb{R}^{m\times n}$ such that

$$A + BF$$

is Hurwitz.

Eigenvalue Assignment: Given $\{\lambda_1, \cdots, \lambda_n\}$, find $F \in \mathbb{R}^{m\times n}$ such that

$$\lambda_i \in \text{eig}(A + BF) \qquad \text{for } i = 1, \cdots, n.$$

Note: A solution to the eigenvalue assignment problem can also solve the stabilization problem.

Question: Is eigenvalue assignment actually harder?

Eigenvalue Assignment

Single-Input Case

Theorem 1.

Suppose $B \in \mathbb{R}^{n\times 1}$. Eigenvalues of $A+BF$ are freely assignable if and only if (A,B) is controllable.

Proof.

1. (Controllable Canonical Form) There exists a T such that

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 0 & I \\ -a_0 & \begin{bmatrix} -a_1 & \cdots & -a_{n-1} \end{bmatrix} \end{bmatrix} \qquad \hat{B} = TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

2. Define $\hat{F} = \begin{bmatrix} \hat{f}_0 & \cdots & \hat{f}_{n-1} \end{bmatrix} \in \mathbb{R}^{1\times n}$. Then

$$\hat{B}\hat{F} = \begin{bmatrix} 0 & 0 \\ \hat{f}_0 & \begin{bmatrix} \hat{f}_1 & \cdots & \hat{f}_{n-1} \end{bmatrix} \end{bmatrix}$$

□

Proof.

- Then

$$\hat{B}\hat{F} = \begin{bmatrix} 0 & & 0 & \\ \hat{f}_0 & [\hat{f}_1 & \cdots & \hat{f}_{n-1}] \end{bmatrix}$$

$$\hat{A} + \hat{B}\hat{F} = \begin{bmatrix} O & & I & \\ -a_0 + \hat{f}_0 & [-a_1 + \hat{f}_1 & \cdots & -a_{n-1} + \hat{f}_{n-1}] \end{bmatrix}$$

- This has the characteristic equation

$$\det\left(sI - (\hat{A} + \hat{B}\hat{F})\right) = s^n + (a_{n-1} - \hat{f}_{n-1})s^{n-1} + \cdots + (a_0 - \hat{f}_0)$$

- Suppose we want eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$. Then define b_i as

$$p(s) = (s - \lambda_1)\cdots(s - \lambda_n) = s^n + b_{n-1}s^{n-1} + \cdots + b_0$$

- Choose $\hat{f}_i = a_i - b_i$.
- Now let $F = \hat{F}T$. Then $A + BF = T^{-1}(\hat{A} + \hat{B}\hat{F})T$

□

Proof.

- Then
$$\det\left(sI-(A+BF)\right) = \det\left(T\left(sI-(\hat{A}+\hat{B}\hat{F})\right)T^{-1}\right) \\ = \det\left(sI-(\hat{A}+\hat{B}\hat{F})\right) \\ = (s-\lambda_1)\cdots(s-\lambda_n)$$
- Hence $A+BF$ has eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.

□

Suppose we want the eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.

1. Find the b_i
2. Choose $\hat{f}_i = a_i - b_i$.
3. Then use $F = \begin{bmatrix}\hat{f}_0 & \cdots & \hat{f}_{n-1}\end{bmatrix} T$.

Conclusion: For Single-Input, controllability implies eigenvalue assignability.

- Requires conversion to controllable canonical form
- Matlab command `acker`.

The multi-input case is harder

Lemma 2.

If (A, B) is controllable, then for any $x_0 \neq 0$, there exists a sequence $\{u_0, u_1, \cdots, u_{n-2}\}$ such that span$\{x_0, x_1, \cdots, x_{n-1}\} = \mathbb{R}^n$, where

$$x_{k+1} = Ax_k + Bu_k \qquad \text{for } k = 0, \cdots, n-1$$

Proof.

For $1 \Rightarrow 2$, we again use proof by contrapositive. We show $(\neg 2 \Rightarrow \neg 1)$.

- Suppose that for any x_0, and any $\{u_0, u_1, \cdots, u_{n-2}\}$, span$\{x_0, \cdots, x_{n-1}\} \neq \mathbb{R}^n$. Then there exists some y such that $y^T x_k = 0$ for any $k = 0, \cdots, n-1$. We can solve explicitly for x_k:

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j$$

Proof.

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j$$

- Let $k = n-1$, and $x_0 = Bu_{n-1}$ for some u_{n-1}. Then for any u

$$y^T x_{n-1} = y^T \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_0 \\ \vdots \\ u_{n-2} \end{bmatrix} = y^T C(A,B) u = 0$$

- Therefore, image$(C(A,B)) \neq \mathbb{R}^n$. Hence (A,B) is not controllable. This proves the lemma.

□

Lemma 3.

Suppose (A,B) is controllable. Then for any nonzero column, $B_1 \in \mathbb{R}^n$, of B, there exists a $F_1 \in \mathbb{R}^{m\times n}$ such that $(A+BF_1, B_1)$ is controllable

Proof.

Suppose (A,B) is controllable. Let $x_0 = B_1$ and apply the previous Lemma to find some input $u_0, \cdots, u_{n-2}$ such that span$\{x_0, \cdots x_{n-1}\} = \mathbb{R}^n$ where

$$x_{k+1} = Ax_k + Bu_k$$

Let $T = \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix}$. Then T is invertible. Let

$$F_1 = \begin{bmatrix} u_0 & \cdots & u_{n-2} \end{bmatrix} T^{-1} = UT^{-1}$$

- This implies $F_1 T = U$ and hence $F_1 x_i = u_i$ for $i = 0, \cdots, n-1$.
- Now expand

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k$$

Eigenvalue Assignment

Multiple-Input Case

Proof.

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BF_1 x_k = [A + BF_1]x_k$$

Which means that $x_k = [A + BF_1]^k x_0$. However, since $x_0 = B_1$, we have

$$\begin{aligned} T &= \begin{bmatrix} x_0 & \cdots & x_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} B_1 & \cdots & (A + BF_1)^{n-1} B_1 \end{bmatrix} \\ &= C(A + BF_1, B_1) \end{aligned}$$

- Since T is invertible, $C(A + BF_1, B_1)$ is full rank and hence $(A + BF_1, B_1)$ is controllable.

□

Eigenvalue Assignment

Multiple-Input Case

Theorem 4.

The eigenvalues of $A+BF$ are freely assignable if and only if (A,B) is controllable.

Proof.

The "only if" direction is clear. Suppose (A,B) is controllable and we want eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$. Let B_1 be the first column of B.

- By Lemma, there exists a F_1 such that $(A+BF_1,B_1)$ is controllable.
- By other Lemma, since the $(A+BF_1,B_1)$ is controllable, the eigenvalues of $(A+BF_1,B_1)$ are assignable. This we can find a F_2 such that $A+BF_1+B_1F_2$ has eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.
- Choose $F=F_1+\begin{bmatrix}F_2\\0\end{bmatrix}$. Then
$$A+BF=A+\begin{bmatrix}B_1 & B_2\end{bmatrix}\left[F_1+\begin{bmatrix}F_2\\0\end{bmatrix}\right]=A+BF_1+B_1F_2$$
has the eigenvalues $\{\lambda_1,\cdots,\lambda_n\}$.

□

Eigenvalue Assignment

Multiple-Input Case

Theorem 5.

The eigenvalues of $A + BF$ are freely assignable if and only if (A, B) is controllable.

Note that the proof was not very constructive: Need to find F_1 and $F_2 \ldots$ **2**

Matlab Commands

- `K=acker(A,B,p)` for 1-D
- `K=place(A,B,p)` for n-D. p is the vector of pole locations.

Theorem 6.

If (A, B) is stabilizable, then there exists a F such that $A + BF$ is Hurwitz.

Proof.

Apply the previous result to the controllability form. □

Conclusion: If (A, B) is stabilizable, then it can be stabilized using *only* static state feedback. $u(t) = Kx(t)$.