Modern Control Systems

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Lecture 09: Observability

Observability

For Static Full-State Feedback, we assume knowledge of the **Full-State**.

- In reality, we only have measurements

$$y_m(t) = C_m x(t)$$

- How to implement our controllers?

Consider a system with no input:

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

Definition 1.

The pair (A, C) is **Observable** on $[0, T]$ if, given $y(t)$ for $t \in [0, T]$, we can find x_0.

Observability

Let $\mathbb{F}(\mathbb{R}^{p_1}, \mathbb{R}^{p_2})$ be the space of functions which map $f : \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$.

Definition 2.

Given (C, A), the flow map, $\Psi_T : \mathbb{R}^p \to \mathbb{F}(\mathbb{R}, \mathbb{R}^p)$ is

$$\Psi_T : x_0 \mapsto Ce^{At}x_0 \qquad t \in [0, T]$$

So $y = \Psi_T x_0$ implies $y(t) = Ce^{At}x_0$.

Proposition 1.

The pair (C, A) is observable if and only if Ψ_T is invertible

$$\ker \Psi_T = 0$$

Observability

Theorem 3.

$$\ker \Psi_T = \ker C \cap \ker CA \cap \ker CA^2 \cap \cdots \cap \ker CA^{n-1} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Proof.

Similar to the Controllability proof: $R_t = \operatorname{image} C(A, B)$ □

Definition 4.

The matrix $O(C, A)$ is called the **Observability Matrix**

$$O(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Unobservable Subspace

Definition 5.

The **Unobservable Subspace** is $N_{CA} = \ker \Psi_T = \ker O(C, A)$.

Theorem 6.

N_{AB} is A-invariant.

Duality

The Controllability and Observability matrices are related

$$O(C, A) = C(A^T, C^T)^T$$
$$C(A, B) = O(B^T, A^T)^T$$

For this reason, the study of controllability and observability are related.

$$\ker O(C, A) = [\operatorname{image} C(A^T, C^T)]^\perp$$
$$\operatorname{image} C(A, B) = [\ker O(B^T, A^T)]^\perp$$

We can investigate observability of (C, A) by studying controllability of (A^T, C^T)

- (C, A) is observable if $\operatorname{image} C(A^T, C^T) = \mathbb{R}^n$

Duality

Definition 7.

For pair (C, A), the **Observability Grammian** is defined as

$$Y = \int_0^\infty e^{A^T s} C^T C e^{As} ds$$

The following seminal result is not surprising

Theorem 8.

For a given pair (C, A), the following are equivalent.

- $\ker Y = 0$
- $\ker \Psi_T = 0$
- $\ker O(C, A) = 0$

If the state is observable, then it is observable arbitrarily fast.

There are several other results which fall out directly.

Theorem 9 (PBH Test).

(C, A) *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}$.

- Again, we can consider only eigenvalues λ.
- No equivalent to Stabilizability?

Observability Form

Theorem 10.

For any pair (C, A)*, there exists an invertible* T *such that*

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \qquad CT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

where the pair $(\tilde{C}_1, \tilde{A}_{11})$ *is observable.*

Invariant Subspace Form

- What is the invariant subspace?

Dissecting the equations (and dropping the tilde), we have

$$\dot{x}_1(t) = A_{11}x_1(t) \qquad \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t)$$
$$y(t) = Cx_1(t)$$

Then we can solve for the output:

$$y(t) = Ce^{A_{11}t}x_1(0)$$

The initial condition $x_2(0)$ does not affect the output in any way!

- $x_2(0) \in \ker \Psi_T$.
- No way to back out $x_2(0)$.

Detectability

The equivalent to stabilizability

Definition 11.

The pair (C, A) is detectable if, when in observability form, $\tilde{A}_{22}$ is Hurwitz.

All unstable states are observable

Theorem 12 (PBH for detectability).

Suppose (C, A) *has observability form*

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \qquad CT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

Then A_{22} *is Hurwitz if and only if*

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}^+$*.*

Observers

Suppose we have designed a controller

$$u(t) = Fx(t)$$

but we can only measure $y(t) = Cx(t)$!

Question: How to find $x(t)$?

- If (C, A) observable, then we can observe $y(t)$ on $t \in [t, t+T]$.
  - But by then its too late!
  - we need $x(t)$ in *real time*!

Observers

Definition 13.

An **Observer**, is an *Artificial Dynamical System* whose output tracks $x(t)$.

Suppose we want to observe the following system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Lets assume the system is state-space

- What are our inputs and output?
- What is the dimension of the system?

Observers

Inputs: $u(t)$ and $y(t)$.
Outputs: Estimate of the state: $\hat{x}(t)$.

Assume the observer has the same dimension as the system

$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)$$
$$\hat{x}(t) = Qz(t) + Ry(t) + Su(t)$$

We want $\lim_{t\to 0} e(t) = \lim_{t\to 0} x(t) - \hat{x}(t) = 0$

- for any u, $z(0)$, and $x(0)$.
- We would also like internal stability, etc.

Observers

System:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Observer:

$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)$$
$$\hat{x}(t) = Qz(t) + Ry(t) + Su(t)$$

What are the dynamics of $x - \hat{x}$?

$$\begin{aligned}
&\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - Q\dot{z}(t) + R\dot{y}(t) + S\dot{u}(t) \\
&= Ax(t) + Bu(t) - Q(Mz(t) + Ny(t) + Pu(t)) + R(C\dot{x}(t) + D\dot{u}(t)) + S\dot{u}(t) \\
&= Ax(t) + Bu(t) - QMz(t) - QN(Cx(t) + Du(t)) - QPu(t) \\
&\qquad\qquad + RC(Ax(t) + Bu(t)) + (S + RD)\dot{u}(t) \\
&= (A + RCA - QNC)e(t) + (AQ + RCAQ - QNCQ - QM)z(t) \\
&+ (A + RCA - QNC)Ry(t) + (B + RCB - QP - QND)u(t) + (S + RD)\dot{u}(t)
\end{aligned}$$

Designing an observer requires that these dynamics are Hurwitz.

Luenberger Observers

Initially, we consider a special class of observers, parameterized by the matrix L

$$\dot{z}(t) = (A + LC)z(t) - Ly(t) + (B + LD)u(t) \tag{1}$$
$$\hat{x}(t) = z(t) \tag{2}$$

In the general formulation, this corresponds to

$$M = A + LC; \qquad N = -L; \qquad P = B + LD;$$
$$Q = I; \qquad R = 0; \qquad S = 0;$$

So in this case $z(t) = \hat{x}(t)$ and $(A + RCA - QNC) = QM = A + LC$.
Furthermore $(A + RCA - QNC)R = 0$ and
$AQ + RCAQ - QNCQ - QM = 0$. Thus the criterion for convergence is
$A + LC$ Hurwitz.

Question Can we choose L such that $A + LC$ is Hurwitz?
Similar to choosing $A + BF$.

Observability

If turns out that controllability and detectability are useful

Theorem 14.

The eigenvalues of $A + LC$ are freely assignable through L if and only if (C, A) is observable.

If we only need $A + LC$ Hurwitz, then the test is easier.

- We only need detectability

Theorem 15.

An observer exists if and only if (C, A) is detectable

Note: Theorem applies to ANY observer, not just Luenberger observers.

Observability

Theorem 16.

An observer exists if and only if (C, A) is detectable

Proof.

We begin with $1) \Rightarrow 2)$. We use proof by contradiction. We show $\not 2) \Rightarrow \not 1)$.

- Suppose (C, A) is not detectable. We will show that for some initial conditions $x(0)$ and $z(0)$, The observer will not converge

$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)$$
$$\hat{x}(t) = Qz(t) + Ry(t) + Su(t)$$

- Convert the system to obervability form where A_{22} is not Hurwitz.

$$\dot{x}_1(t) = A_{11}x_1(t)$$
$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t)$$
$$y(t) = Cx_1(t)$$

□

Observability

Proof.

- Choose $x_1(0) = 0$ and $x_2(0)$ to be an eigenvector of A_{22} with associated eigenvalue λ having positive real part.
- Then $x_1(t) = e^{A_{11}t}x_1(0) = 0$ for all $t > 0$.
- Then
$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) = A_{22}x_2(t).$$
Hence $x_2(t) = e^{A_{22}t}x_2(0) = x_2(0)e^{\lambda t}$. Thus $\lim_{t\to\infty} x_2(t) = \infty$.
- However, $y(t) = Cx_1(t) = 0$ for all $t > 0$.
- For any observer, choose $z(0) = 0$ and $u(t) = 0$. Then
$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t) = Mz(t)$$
Hence $z(t) = e^{Mt}z(0) = 0$ for all $t > 0$ and $\hat{x}(t) = 0$ for all $t > 0$.
- We conclude that $\lim_{t\to\infty} e(t) = \lim_{t\to\infty} x(t) - \hat{x}(t) = \infty$

□

Observability

Theorem 17.

An observer exists if and only if (C, A) is detectable

Proof.

Next we prove that $2) \Rightarrow 1)$. We do this directly by constructing the observer.

- If (C, A) is detectable, then there exists a L such that $A + LC$ is Hurwitz.
- Choose the Luenberger observer

$$\begin{aligned}\dot{z}(t) &= (A + LC)z(t) - Ly(t) + (B + LD)u(t) \\ \hat{x}(t) &= z(t)\end{aligned}$$

- Referencing previous slide, $A + RCA - QNC = QM = A + LC$ and $B + RCB - QP - QND = 0$ and $S + RD = 0$
- Then the error dynamics become

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A + LC)e(t)$$

- Which has solution $\lim_{t\to\infty} e^{(A+LC)t}e(0) = 0$.
- Thus the observer converges.

Review: Luenberger Observer

$$\dot{z}(t) = (A + LC)z(t) - Ly(t) + (B + LD)u(t) \tag{3}$$
$$\hat{x}(t) = z(t) \tag{4}$$

Theorem 18.

The eigenvalues of $A + LC$ are freely assignable through L if and only if (C, A) is observable.

Theorem 19.

An observer exists if and only if (C, A) is detectable

Dynamic Coupling

Question: How to compute L?

- The eigenvalues of $A + LC$ and $(A + LC)^T = A^T + C^T L^T$ are the same.
- This is the same problem as controller design!

Answer: Choose a vector of eigenvalues E.

- `L = place`$(A^T, C^T, E)^T$

So now we know how to design an Luenberger observer.

- Also called an estimator

The error dynamics will be dictated by the eigenvalues of $A + LC$.

- For fast convergence, chose very negative eigenvalues.
- generally a good idea for the observer to converge faster than the plant.

Observer-Based Controllers

Summary: What do we know?

- How to design a controller which uses the full state.
- How to design an observer which converges to the full state.

Question: Is the combined system stable?

- We know the error dynamics converge.
- Lets look at the coupled dynamics.

Proposition 2.

The system defined by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
u(t) &= F\hat{x}(t) \\
\dot{\hat{x}}(t) &= (A + LC + BF + LDF)\,\hat{x}(t) - Ly(t)
\end{aligned}$$

has eigenvalues equal to that of $A + LC$ and $A + BF$.

Note we have reduced the dependence on $u(t)$.

Observer-Based Controllers

The proof is relatively easy

Proof.

The state dynamics are

$$\dot{x}(t) = Ax(t) + BF\hat{x}(t)$$

Rewrite the estimation dynamics as

$$\begin{aligned}
\dot{\hat{x}}(t) &= (A + LC + BF + LDF)\,\hat{x}(t) - Ly(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,F\hat{x}(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,u(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + Bu(t) - LCx(t) \\
&= (A + LC + BF)\,\hat{x}(t) - LCx(t)
\end{aligned}$$

In state-space form, we get

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + LC + BF \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$\square$

Proof.

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

Use the similarity transform $T = T^{-1} = \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}$.

$$\begin{aligned}T\bar{A}T^{-1} &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A & BF\\ -LC & A+LC+BF\end{bmatrix}\begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\\ &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A+BF & -BF\\ A+BF & -(A+LC+BF)\end{bmatrix}\\ &= \begin{bmatrix}A+BF & -BF\\ 0 & A+LC\end{bmatrix}\end{aligned}$$

which has eigenvalues $A + LC$ and $A + BF$. $\square$