Modern Optimal Control

Matthew M. Peet
Arizona State University

Lecture 12: Small Gain Theorem

A algebra is a vector space with a distributive multiplication operator.

Definition 1.

A **Banach Algebra**, X, is a Banach space with an associated mapping $(\cdot) : X \times X \to X$ such that

1. **Identity:** There exists some $I \in X$ such that
$$F \cdot I = I \cdot F = F$$
for all $F \in X$.
2. **Distributivity:** $F \cdot (G \cdot H) = (F \cdot G) \cdot H$ for all $F, G, H \in X$
3. **Associativity:** $F \cdot (G + H) = F \cdot G + F \cdot H$ for all $F, G, H \in X$.
4. $F \cdot (\alpha G) = (\alpha F) \cdot G$ for all $F, G \in X$ and $\alpha \in \mathbb{R}$.
5. **Submultiplicative Inequality:**
$$\|F \cdot G\| \leq \|F\| \|G\|$$

Some algebras have extra properties

- **Inverse Property (Group):** For any $F \in X$, there exists a $F^{-1} \in X$ such that
$$F \cdot F^{-1} = F^{-1} \cdot F = I$$
- **Commutative Algebra, Abelian Group:** $F \cdot G = G \cdot F$

Square Matrices:

- Matrix multiplication using the $\bar{\sigma}$ norm.

Vectors:

- Pointwise addition/multiplication **only**.

Linear Operators: $\mathcal{L}(X)$

- Using the composition operation and induced norm.

Some elements of a Banach Algebra may have an inverse.

Definition 2.

For $J \in X$, where X is a Banach Algebra, we say K is the **inverse** of J if $K \in X$ and

$$J \cdot K = K \cdot J = I$$

If such a K exists, we say J is **invertible**.

Note that if the inverse exists, it is unique

Observation: The feedback interconnection yields

$$y = (I + GK)^{-1} Gu$$

Question: Given G and K, how to tell whether $(I + GK)$ is invertible?

- How to tell whether anything is invertible?

Answer: Spectral Theory

- When is $\lambda I - A$ invertible?

Small Gain Theorem

The simplest form of spectral theory.

Theorem 3 (Small Gain Theorem).

Suppose B is a Banach Algebra and $Q \in B$. If $\|Q\| < 1$, then $(I-Q)^{-1}$ exists and furthermore

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

Clearly holds for $B = \mathbb{R}$ since

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = (1-r)^{-1}$$

Small Gain Theorem

Proof.

Relatively Simple: Show $\sum_{k=0}^{\infty} Q^k$ converges and that it is the inverse.

- To show that the sequence $T_i = \sum_{k=0}^{i} Q^k$ converges, we show it is Cauchy.
- Suppose $m > n$. Then

$$\|T_m - T_n\| = \Big\| \sum_{k=n+1}^{m} Q^k \Big\| \le \sum_{k=n+1}^{m} \|Q^k\| \quad \text{Triangle Inequality}$$

$$\le \sum_{k=n+1}^{m} \|Q\|^k \quad \text{Submultiplicative Inequality}$$

$$= \|Q\|^{n+1} \sum_{k=0}^{m-n-1} \|Q\|^k$$

$$\le \|Q\|^{n+1} \sum_{k=0}^{\infty} \|Q\|^k$$

$$= \|Q\|^{n+1} \frac{1}{1 - \|Q\|} \quad \text{Scalar Power Series}$$

Small Gain Theorem

Proof.

- Thus for $m > n$
$$\lim_{n\to\infty} \|T_m - T_n\| = \lim_{n\to\infty} \|Q\|^{n+1} \frac{1}{1-\|Q\|} = 0$$
since $\|Q\| < 1$.
- Since the sequence is Cauchy and the space Banach, $\sum_{k=0}^{\infty} Q^k \in B$

We have shown that $\sum_{k=0}^{\infty} Q^k \in B$. Now we show that it is the inverse.

- Start by showing it is a right inverse
$$\begin{aligned}(I-Q)\sum_{k=0}^{\infty} Q^k &= \sum_{k=0}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k \\ &= Q^0 + \sum_{k=1}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k = I\end{aligned}$$
- Showing $\sum_{k=0}^{\infty} Q^k (I-Q) = I$ is identical.
- Thus $(I-Q)^{-1}$ exists, so $(I-Q)$ is invertible.

Small Gain Theorem

Recall the feedback interconnection:

$$y = (I + GK)^{-1} G u$$

Question: When is $(I + GK)^{-1} G \in \mathcal{L}(L_\infty)$?

Answer: When $\|GK\| < 1$.

- Using Banach Algebra of Composition

$$\|GK\| \le \|G\| \|K\|$$

- We can require $\|K\| < \frac{1}{\|G\|}$ or $\|K\| < 1$ and $\|G\| \le 1$.
- Can also express as $\int_0^\infty \|Ce^{As}B\| ds \le 1$

Note: Small Gain Theorem works for **ANY** Banach space (unusual).

- For most results we need a Hilbert space.

Take the Banach Algebra of square matrices. $(\mathbb{R}^{n\times n}, \|\cdot\| = \bar{\sigma}(\cdot))$.

- Let $Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$, with norm $\bar{\sigma}(Q) = \frac{1}{2}$.
- By small gain $(I-Q)^{-1}$ exists. Further

$$(I-Q)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \sum_{k=0}^{\infty} Q^k$$

- Now

$$Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 0 & \frac{1}{8} \\ \frac{1}{8} & 0 \end{bmatrix}, \quad Q^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$$

- We conclude that

$$Q^{k_{\text{even}}} = \begin{bmatrix} \frac{1}{4^k} & 0 \\ 0 & \frac{1}{4^k} \end{bmatrix} = \frac{1}{4^k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^{k_{\text{odd}}} = \begin{bmatrix} 0 & \frac{1}{2}\frac{1}{4^k} \\ \frac{1}{2}\frac{1}{4^k} & 0 \end{bmatrix} = \frac{1}{4^k}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

- Thus

$$\sum_{k=0}^{\infty} Q^k = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{1}{4^k} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \frac{1}{1-\frac{1}{4}} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \frac{4}{3} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

Small Gain Theorem

Example

Unfortunately, the small gain theorem is conservative.

- Let

$$Q = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}$$

- Then $\bar{\sigma}(Q) = 10$, yet

$$(I - Q)^{-1} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}^{-1}$$

- Furthermore, $(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k$.

Spectral Theorem

An extension of the concept of eigenvalues.

- with important differences.

Definition 4.

Let B be a Banach Space and $M \in \mathcal{L}(B)$. The **Spectrum** of M is:

$$\sigma(M) := \{\lambda \in \mathbb{C} \;:\; (\lambda I - M) \text{ is not invertible in } \mathcal{L}(B)\}$$

The **Spectral Radius** is

$$\rho(M) := \max\{|\lambda| \;:\; \lambda \in \sigma(M)\}$$

Fact: $\sigma(M)$ is non-empty and closed.

- all you can say.

Consider a multiplication operator.

$$M : u(t) \mapsto e^{-t}u(t)$$

Theorem 5.

$M \in \mathcal{L}(L_\infty[0,\infty))$ and $\sigma(M) = [0,1]$.

Proof.

First we show that $[0,1] \subset \sigma(M)$.

- Since $((\lambda I - M)u)(t) = (\lambda - e^{-t})\, u(t)$, we can construct the inverse

$$\left((\lambda I - M)^{-1} u\right)(t) = \frac{1}{\lambda - e^{-t}} u(t)$$

- If $\lambda \in [0,1]$, then $\lim_{t \to -\log(\lambda)} \frac{1}{\lambda - e^{-t}} = \infty$.
- Therefore, $(\lambda I - M)^{-1}$ is unbounded.
- Thus $[0,1] \subset \sigma(M)$.

□

Spectral Theorem

Proof.

Now we show that if $\lambda \notin [0,1]$, then $\lambda \notin \sigma(M)$.

- If $\lambda < 0$, then

$$\left|\left((\lambda I - M)^{-1}u\right)(t)\right| = \frac{1}{\lambda - e^{-t}}u(t) \leq \frac{1}{|\lambda|}|u(t)|$$

- If $\lambda > 1$, then

$$\left|\left((\lambda I - M)^{-1}u\right)(t)\right| = \frac{1}{\lambda - e^{-t}}u(t) \leq \frac{1}{|\lambda + 1|}|u(t)|$$

- Thus $[0,1] = \sigma(M)$.

□

However, M has no eigenvalues. If $Mx = \lambda x$, then

$$e^{-t}x(t) = \lambda x(t) \qquad \forall t$$

which implies that $x(t) = 0$.

- Thus M has no eigenvalues

Don't assume eigenvalues exist.

Spectral Theorem

However, some properties still hold.

Theorem 6.

For any $M \in \mathcal{L}(B)$, $\rho(M) \leq \|M\|$.

Proof.

We will show that if $|\lambda| > \|M\|$, then $\lambda \notin \sigma(M)$.

- Suppose $|\lambda| > \|M\|$. Define $Q = \frac{M}{\lambda}$. Then

$$\|Q\| = \frac{1}{|\lambda|}\|M\| < 1$$

- Then by the small-gain theorem, $(I - Q)$ is invertible with

$$(I - Q)^{-1} = \left(I - \frac{M}{\lambda}\right)^{-1} = \sum_{k=0}^{\infty} \frac{M^k}{\lambda^k}$$

- Thus $(\lambda I - M)^{-1} = \lambda^{-1}\left(I - \frac{M}{\lambda}\right)^{-1}$ exists.
- Thus $\lambda \notin \sigma(M)$.
- Therefore, if $\lambda \in \sigma(M)$, $|\lambda| \leq \|M\|$.
- Thus $\rho(M) \leq \|M\|$.

Spectral Theorem

Theorem 7.

If $M \in \mathcal{L}(B)$, *then* $\rho(M) = \lim_{k\to\infty} \|M^k\|^{1/k}$.

No Proof

Spectral Theorem

For the Banach Algebra of linear operators, order of operations doesn't matter.

Lemma 8.

For $M \in \mathcal{L}(U,V)$ and $Q \in \mathcal{L}(V,U)$, the following are equivalent.

1. $I_V - MQ$ *is invertible.*
2. $I_U - QM$ *is invertible.*

Proof.

Start with $1 \Rightarrow 2$. Suppose $(I - MQ)^{-1}$ exists. Then we propose the inverse $(I - QM)^{-1} = (I + Q(I - MQ)^{-1}M$. Then

$$
\begin{aligned}
&(I + Q(I - MQ)^{-1}M)(I - QM) \\
&\qquad = I(I - QM) + Q(I - MQ)^{-1}M(I - QM) \\
&\qquad = I - QM + Q(I - MQ)^{-1}M - Q(I - MQ)^{-1}MQM \\
&\qquad = I + Q\left[-I + (I - MQ)^{-1} - (I - MQ)^{-1}MQ\right]M \\
&\qquad = I + Q\left[-I + (I - MQ)^{-1}(I - MQ)\right]M \\
&\qquad = I + Q\left[-I + I\right]M = I
\end{aligned}
$$

Spectral Theorem

continued.

$$(I + Q(I - MQ)^{-1}M)(I - QM) = I$$

- Thus $(I - QM)^{-1} = I + Q(I - MQ)^{-1}M$

We can show in a similar way that $(I - MQ)^{-1} = I + M(I - QM)^{-1}Q$. □

Spectral Theorem

This leads directly to

Proposition 1.

For $M \in \mathcal{L}(U,V)$, $Q \in \mathcal{L}(V,U)$, $\sigma(MQ) \cup \{0\} = \sigma(QM) \cup \{0\}$.

Proof.

Suppose $\lambda \neq 0$. Then

$$\begin{aligned}
\lambda \in \sigma(MQ) \Leftrightarrow &\quad \lambda I - MQ \quad \text{is invertible} \\
\Leftrightarrow &\quad I - \frac{MQ}{\lambda} \quad \text{is invertible} \\
\Leftrightarrow &\quad I - \frac{QM}{\lambda} \quad \text{is invertible by previous lemma} \\
\Leftrightarrow &\quad \lambda I - QM \quad \text{is invertible} \\
\Leftrightarrow &\quad \lambda \in \sigma(QM)
\end{aligned}$$

□