Modern Control Systems

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Lecture 13: The joy of Hilbert Space

Recall: Hilbert Spaces

Definition 1.

An inner product space which is complete in the norm $\|x\|^2 = \langle x, x \rangle$ is called a **Hilbert Space**.

Example: Define the following inner product on $L_2[0, \infty)$:

$$\langle x, y \rangle_{L_2} := \int_0^\infty x^T(s) y(s) ds$$

Then

$$\|x\|_{L_2}^2 = \int_0^\infty \|x(s)\|^2 ds$$

And since L_2 is complete in this norm, $L_2[0, \infty)$ is a Hilbert Space.

The previous lecture only required a Banach Algebra. From now on we will only consider linear operators on Hilbert spaces.

- $\mathcal{L}(H)$ is still a Banach space, though.

Operators $\mathcal{L}(H)$ acting on a Hilbert space H, have **most** of the properties of matrices

Property 1: Adjoint

Definition 2.

Suppose V, Z are Hilbert spaces. For any $F \in \mathcal{L}(V, Z)$, we say that $F^* \in \mathcal{L}(Z, V)$ is the **adjoint** of F if

$$\langle z, Fv \rangle_Z = \langle F^* z, v \rangle_V$$

for all $z \in Z$ and $v \in V$.

Proposition 1.

Suppose U, V are Hilbert. Then for any $F \in \mathcal{L}(U,V)$, F^* exists, is unique and

$$\|F\| = \|F^*\| = \sqrt{\|F^*F\|}$$

Examples

- Matrix Transpose
- The convolution operator $F \in \mathcal{L}(L_2)$

$$(Fu)(t) = \int_0^t f(t-s)u(s)ds$$

Then

$$(F^*u)(t) = \int_0^t f^*(s-t)u(s)ds$$

Self-Adjoint Operators

There are "symmetric" operators.

Definition 3.

An operator is **Self-Adjoint** if $F = F^*$.

- If $F = F^*$, then $\langle v, Fv \rangle \in \mathbb{R}$.
- Proof?

There are also "Positive" operators (related to Passivity, but not the same)

Definition 4.

A self-adjoint operator, $F = F^* \in \mathcal{L}(H)$ is **Positive Definite** (PD, denoted $F > 0$) if there exists some $\epsilon > 0$ such that

$$\langle v, Fv \rangle \geq \epsilon \|v\|^2$$

- Note that $\langle v, Fv \rangle > 0$ for all v is not equivalent.

Definition 5.

A self-adjoint operator, $F = F^* \in \mathcal{L}(H)$ is **Positive Semidefinite** (PSD, denoted $F \geq 0$) if

$$\langle v, Fv \rangle \geq 0$$

- F is Positive Definite if and only if $F - \epsilon I \geq 0$ for some $\epsilon > 0$

Property 2: Square Root

Proposition 2.

1. *If $F \geq 0$ is PSD, then there exists some PSD operator $F^{1/2} \geq 0$ such that*

$$F = F^{\frac{1}{2}} F^{\frac{1}{2}}$$

2. *If $F > 0$ is PD, then there exists some PD operator $F^{1/2} > 0$ such that*

$$F = F^{\frac{1}{2}} F^{\frac{1}{2}}$$

Proposition 3.

If $M \in \mathcal{L}(V)$ and $M = M^*$, then $\rho(M) = \|M\|$.

Corollary 6.

If $M \in \mathcal{L}(U, V)$, then

$$\rho(M^* M) = \|M\|^2$$

Positive Matrices have positive spectra.

Proposition 4.

- *If $M = M^*$, then $\sigma(M) \subset \mathbb{R}$*
- *If $M = M^* \geq 0$, then $\sigma(M) \subset \mathbb{R}^+$*

Definition 7.

An Operator $F \in \mathcal{L}(U,V)$ is **Isometric** if

$$F^* F = I_U$$

- If $F \in \mathcal{L}(U,V)$ is isometric, then for any $x, y \in U$

$$\langle Fx, Fy \rangle_V = \langle x, y \rangle_U$$

- If F is isometric, then $\|F\| = 1$. Proof:

$$\sup \frac{\|Fx\|^2}{\|x\|^2} = \sup \frac{\langle Fx, Fx \rangle}{\langle x, x \rangle} = \sup \frac{\langle x, x \rangle}{\langle x, x \rangle} = 1$$

Definition 8.

An operator $F \in \mathcal{L}(X, Y)$ is **Unitary** if

$$F^* = F^{-1}$$

- A unitary operator is obviously isometric since

$$F^* F = F^{-1} F = I$$

- If $F \in \mathcal{L}(X, Y)$ is unitary, then X and Z are isomorphic
  - There always exists a unitary operator between isomorphic spaces.