# Modern Control Systems

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Lecture 14: Operators on Signal Space

Now we are ready to define the Laplace transform.

**Recall:** $L_2(-\infty, \infty)$ is the space of functions with inner product given by

$$\langle u, y\rangle_{L_2} = \int_{-\infty}^{\infty} u(t)^* y(t) dt$$

Now, we propose a new signal space in the frequency domain: $\hat{L}_2$

**Definition 1.**

$\hat{L}_2$ is the inner-product space of functions $\hat{f} : \mathbb{R} \to \mathbb{C}^n$ with form $\hat{f}(\imath\omega)$ and inner product

$$\langle \hat{u}, \hat{y}\rangle_{\hat{L}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\imath\omega)^* \hat{v}(\imath\omega) d\omega$$

$\hat{L}_2$ inherits the norm

$$\|\hat{u}\|_{\hat{L}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|^2 d\omega$$

**Note:** The use of " $\hat{\cdot}$ " notation will refer to frequency-domain spaces, signals and operators.

# The Fourier Transform

## Definition 2.

For any function $u : \imath\mathbb{R} \to \mathbb{R}^n$, we define the **Fourier Transform** of $u$: $\phi u$ by

$$\hat{u} = (\phi u)(\imath\omega) = \int_{-\infty}^{\infty} u(t) e^{-\imath\omega t} dt$$

- Note that we neglected the signal space
- On $L_2$, we have $\phi u(\imath\omega) = \langle u, e^{-\imath\omega t} \rangle_{L_2}$

## Theorem 3.

- *If $u \in L_1$, the $(\phi u)(\imath\omega)$ is well defined for all $\omega \in \mathbb{R}$*
- *If $u \in L_2$, then*

$$\lim_{T \to \infty} \int_{-T}^{T} u(t) e^{-\imath\omega t} dt$$

*exists for almost all $\omega$.*

  - *When limit does not exist, define $(\phi u)(\imath\omega) = 0$*

# The "Inverse Fourier Transform"

**Note:** We have not shown that $\phi$ has an inverse (or any other properties).

## Definition 4.

Given a function $\hat{u} : \mathbb{R} \to \mathbb{C}^n$, we propose the operator $\phi^{-1}$

$$(\phi^{-1}\hat{u})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\imath\omega)e^{\imath\omega t}d\omega$$

- In this case $(\phi^{-1}\hat{u})(t) = \langle \hat{u}, e^{\imath\omega t}\rangle_{\hat{L}_2}$
- If $\hat{u} \in \hat{L}_2(\imath\mathbb{R})$, then $(\phi^{-1}u)(t)$ exists for almost all $t$.

**Invertible?**

- If $(\phi u)(\imath\omega)$ exists for almost all $\omega$, then
$$(\phi^{-1}\phi u)(t) = u(t)$$
for almost all $t$.
- If $(\phi^{-1}\hat{u})(t)$ exists for almost all $t$, then
$$(\phi\phi^{-1}\hat{u})(\imath\omega) = u(\imath\omega)$$
for almost all $\omega$.

# The Plancherel Theorem

The Fourier Transform and its inverse are Unitary.

**Theorem 5 (Plancherel).**

1. $\phi : L_2(-\infty, \infty) \to \hat{L}_2(\imath\mathbb{R})$ *and*

$$\langle u, v\rangle_{L_2} = \langle \phi u, \phi v\rangle_{\hat{L}_2} \qquad \textit{for all} \quad u, v \in L_2.$$

2. $\phi^{-1} : \hat{L}_2(\imath\mathbb{R}) \to L_2(-\infty, \infty)$ *and*

$$\langle \hat{u}, \hat{v}\rangle_{\hat{L}_2} = \langle \phi^{-1}\hat{u}, \phi^{-1}\hat{v}\rangle_{L_2} \qquad \textit{for all} \quad \hat{u}, \hat{v} \in \hat{L}_2.$$

Unitary because

$$\langle u, \phi^*\phi v\rangle_{L_2} = \langle \phi u, \phi v\rangle_{\hat{L}_2} = \langle u, v\rangle_{L_2}$$

for all $u, v$, which implies $\phi^*\phi = I$.

- Shows that $\phi^{-1} = \phi^*$ is the Inverse of the Fourier Transform

Now we know $L_2$ and $\hat{L}_2$ are isomorphic.

# The Fourier System

Let $y = Gu$.

- Then $y = \phi^{-1}\phi G\phi^{-1}\phi u$
- OR $\hat{y} = (\phi G\phi^{-1})\hat{u}$
- We'll return to $\phi G\phi^{-1}$ shortly

Because $L_2$ and $\hat{L}_2$ are isomorphic, $\hat{L}_2$ are the coordinates of $u$ in the Fourier basis.

- $\hat{L}_2(\imath\omega)$ is the coordinate of basis $e^{-\imath\omega t}$.

The problem with operators on $\hat{L}_2$ is they are not always **Causal**.

# Laplace Transform

Now consider the space $L_2[0, \infty)$.

**Definition 6.**

Given $u \in L_2[0, \infty)$, the Laplace Transform of $u$ is $\hat{u} = \Lambda u$, where

$$\hat{u}(s) = (\Lambda u)(s) = \lim_{T \to \infty} \int_0^T u(t) e^{-st} dt$$

if this limit exists.

Note that for $u \in L_2[0, \infty)$, $\Lambda u = \phi u$.

- Laplace transform acts on a subspace of $L_2(-\infty, \infty)$
- Laplace and Fourier transforms coincide on the imaginary axis.
- Thus the image of the Laplace transform is "smaller" than the image of $\phi$.
  - Speaking of which: What is the image?
  - Its a bit more complicated.....

# Analytic Functions

Let $u \in L_2[0,\infty)$. Suppose $\text{Re}(s) > 0$.

- Then $e^{-st} \in L_2$ - a basis function
- Then
$$\hat{u}(s) = (\Lambda u)(s) = \int_0^\infty e^{-st} u(t) dt = \langle e^{-st}, u \rangle_{L_2} < \infty$$
- $(\Lambda u)(s)$ is well-defined everywhere in the right-half-plane (RHP).

**Definition 7.**

In complex analysis, a function is **analytic** if it is continuous and bounded.

- More generally, a function is analytic if the Taylor series converges everywhere in the domain.
- $\text{image}\,\Lambda$ is a subset of analytic functions bounded on the right-half-plane.
  - Note we didn't prove continuous.

# To the point

## Definition 8.

A function $\hat{u} : \bar{\mathbb{C}}^+ \to \mathbb{C}^n$ is in $H_2$ if

1. $\hat{u}(s)$ is analytic on the Open RHP (denoted $\mathbb{C}^+$)
2. For almost every real $\omega$,

$$\lim_{\sigma \to 0^+} \hat{u}(\sigma + \imath\omega) = \hat{u}(\imath\omega)$$

   - Which means continuous to the imaginary axis

3. $$\sup_{\sigma \geq 0} \int_{-\infty}^{\infty} \|\hat{u}(\sigma + \imath\omega)\|_2^2 < \infty$$

   - Which means bounded on every vertical line.

# Maximum Modulus Principle

**Theorem 9 (Maximum Modulus).**

*An analytic function cannot obtain its extrema in the interior of the domain.*

Hence if $\hat{u}$ satisfies 1) and 2), then

$$\begin{aligned}\sup_{\sigma\geq 0}\int_{-\infty}^{\infty}\|\hat{u}(\sigma+\imath\omega)\|_2^2 &= \int_{-\infty}^{\infty}\|\hat{u}(\imath\omega)\|_2^2 d\omega \\ &= \|\hat{u}\|_{\hat{L}_2} = \|\phi u\|_{\hat{L}_2} \\ &= \|u\|_{L_2}\end{aligned}$$

Thus we equip $H_2$ with the $\hat{L}_2$ norm and inner product

$$\|\hat{u}\|_{H_2} = \int_{-\infty}^{\infty}\|\hat{u}(\imath\omega)\|_2^2 d\omega, \qquad \langle \hat{u}, \hat{y}\rangle_{H_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(\imath\omega)^*\hat{v}(\imath\omega)d\omega$$

- This is a valid inner product because $H_2$ is isomorphic to the image $\phi L_2[0,\infty)$, which is a subspace of $\hat{L}_2$ (Paley-Wiener, next slide).

# Paley-Wiener

## Theorem 10.

1. *If $u \in L_2[0,\infty)$, then $\Lambda u \in H_2$.*
2. *If $\hat{u} \in H_2$, then there exists a $u \in L_2[0,\infty)$ such that $\hat{u} = \Lambda u$ (**Onto**).*

- Shows that $H_2$ is exactly the image of $\Lambda$ on $L_2[0,\infty)$
- Is the map invertible?

## Definition 11.

The inverse of the Laplace transform,$\Lambda^{-1} : H_2 \to L_2[0,\infty)$ is

$$u(t) = (\Lambda^{-1}\hat{u})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sigma t} \cdot e^{\imath\omega t}\hat{u}(\sigma + \imath\omega)d\omega$$

where $\sigma$ can be any real number.

# Corollary: $H_2$ is Hilbert

## Corollary 12.

*For $\hat{u}, \hat{v} \in H_2$, if $\hat{u}(\imath\omega) = \hat{v}(\imath\omega)$ for all $\omega \in \mathbb{R}$, then $\hat{u}(s) = \hat{u}(s)$ for all $s \in \bar{C}^+$.*

- Only the imaginary axis matters.

## Proof.

Suppose $\hat{u}(\imath\omega) = \hat{v}(\imath\omega)$ for all $\omega$.

- By Paley-Wiener, there exists $u, v \in L_2[0, \infty)$ such that

$$\hat{u} = \Lambda u, \qquad \hat{v} = \Lambda v$$

- Thus $\hat{u} - \hat{v} = \Lambda(u - v)$
- Since the Laplace and Fourier Transforms coincide on the imaginary axis,

$$0 = \hat{u}(\imath\omega) - \hat{v}(\imath\omega) = [\phi(u - v)]\,(\imath\omega)$$

- Thus $\phi(u - v) = 0$
- By Plancherel, $\|u - v\|_{L_2} = 0$. Thus $u(t) = v(t)$ for almost all $t$.
- Thus $\Lambda(u - v) = 0$ and so $\hat{u} = \hat{v}$.

# Corollary

A Subspace

- If $\hat{u}(s) \in H_2$, then $\hat{u}(\imath\omega) \in \hat{L}_2$
- If $\hat{u}(\imath\omega) \in \hat{L}_2$, then $\hat{u}(s) \in H_2$ if and only if it has an analytic continuation to the right-half-plane.

Thus the corollary ensures that $\|\hat{u}\|_{H_2} = 0$ if and only if $\hat{u} = 0$.

- Thus the norm is valid.
- The inner product is also valid.

**Lemma 13.**

*$H_2$ is a closed subspace of $\hat{L}_2$*

- *Thus $H_2$ is a Hilbert space.*

# Corollary

**Lemma 14.**

$$\langle \Lambda u, \Lambda y \rangle_{H_2} = \langle \phi u, \phi y \rangle_{\hat{L}_2} = \langle u, y \rangle_{L_2}$$

- *Thus* $\Lambda$ *is unitary.*
- $L_2[0, \infty)$ *and* $H_2$ *are isomorphic.*

# Example

Let

$$\hat{u}(s) = \frac{e^s}{s+1}$$

- $\hat{u}$ is in $\hat{L}_2$ since

$$\left\| \frac{e^{\imath\omega}}{\imath\omega + 1} \right\|_{\hat{L}_2} = \frac{1}{2}$$

- $\phi^{-1}\hat{u} \notin L_2[0,\infty)$ since

$$\phi^{-1}\hat{u} = u(t) = \begin{cases} e^{-t+1} & t \geq -1 \\ 0 & \text{otherwise} \end{cases}$$

- Thus $\hat{u} \notin H_2$.

# Perp Spaces

The Perp spaces of $L_2[0,\infty)$ and $H_2$ are also related.
**Recall:**

## Definition 15.

The orthogonal complement (perp space) of a closed subspace $S$, of a Hilbert Space, $X$ is

$$S^\perp := \{v \in X \,:\, \langle v, s\rangle = 0 \quad \text{for all } s \in S\}$$

$L_2[0,\infty)^\perp$**:**

## Proposition 1.

$$L_2[0,\infty)^\perp = L_2(-\infty, 0]$$

# Perp Spaces

For $x \in L_2(-\infty, 0]$, we have the left-sided Laplace transform

**Definition 16.**

$$(\Lambda_- u)(s) = \int_{-\infty}^{0} e^{-st} u(t) dt$$

Then $H_2^\perp = \text{image}(\Lambda_-)$ over $L_2(-\infty, 0]$.