# Modern Control Systems

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Lecture 15: Linear Causal Time-Invariant Operators

Because $L_2(-\infty, \infty)$ and $\hat{L}_2$ are isomorphic, so are the sets of operators $\mathcal{L}(L_2)$ and $\mathcal{L}(\hat{L}_2)$.

- Prove using the map $M \mapsto \phi M \phi^{-1}$ and $\hat{M} \mapsto \phi^{-1} \hat{M} \phi$.

How to parameterize $\mathcal{L}(L_2)$?

# $\hat{L}_\infty$ and Multiplication Operators

We now define the new space

**Definition 1.**

Let $\hat{L}_\infty(\imath\mathbb{R})$ be the space of matrix-valued functions $\hat{G} : \imath\mathbb{R} \to \mathbb{C}^{m\times n}$ such that

$$\|\hat{G}\|_{\hat{L}_\infty} = \text{ess} \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\hat{G}(\imath\omega)) < \infty$$

Every element of $\hat{L}_\infty$ defines a multiplication operator.

**Definition 2.**

Given $\hat{G} \in \hat{L}_\infty(\imath\mathbb{R})$, define

$$(M_{\hat{G}}\hat{u})(\imath\omega) = \hat{G}(\imath\omega)\hat{u}(\imath\omega)$$

# $\hat{L}_\infty$

Every multiplication operator defined by $\hat{L}_\infty$ is a bounded linear operator.

## Proposition 1.

*For any $\hat{G} \in \hat{L}_\infty(\imath\mathbb{R})$, $M_{\hat{G}} \in \mathcal{L}(\hat{L}_2)$. Furthermore*

$$\|M_{\hat{G}}\|_{\mathcal{L}(\hat{L}_2)} = \|\hat{G}\|_{\hat{L}_\infty}$$

## Proof.

Sufficiency is easy.

$$\begin{aligned}
\|M_{\hat{G}}\hat{u}\|_{\hat{L}_2}^2 &= \int_{-\infty}^{\infty} (M_{\hat{G}}\hat{u})(\imath\omega)^*(M_{\hat{G}}\hat{u})(\imath\omega)d\omega \\
&= \int_{-\infty}^{\infty} \hat{u}(\imath\omega)^*\hat{G}(\imath\omega)^*\hat{G}(\imath\omega)\hat{u}(\imath\omega)d\omega \\
&\leq \sup_{\omega}\|\hat{G}(\imath\omega)\|^2 \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|^2 d\omega \\
&= \|\hat{G}\|_{\hat{L}_\infty}^2 \|\hat{u}\|_{\hat{L}_2}^2
\end{aligned}$$

# $\hat{L}_\infty$ and $\mathcal{L}(L_2)$

Because the Fourier Transform is unitary, $M_{\hat{G}}$ also defines an operator in $\mathcal{L}(\hat{L}_2)$ with equivalent norm.

- If $G = \phi^{-1} M_{\hat{G}} \phi$, then

$$\|G\|_{\mathcal{L}(L_2)} = \|M_{\hat{G}}\|_{\mathcal{L}(\hat{L}_2)} = \|\hat{G}\|_{\hat{L}_\infty}.$$

**Question:** For every $G \in \mathcal{L}(L_2)$ does there exist some $\hat{G} \in \hat{L}_\infty$ such that

$$G = \phi^{-1} M_{\hat{G}} \phi?$$

# Time-Invariant Systems

To answer the previous question affirmatively, we must consider a subspace of linear operators.

## Definition 3.

Define the shift operator $S_\tau : L_2(-\infty, \infty) \to L_2(-\infty, \infty)$ by

$$(S_\tau u)(t) = u(t - \tau)$$

- Also called the delay operator
- Well defined on both $L_2(-\infty, \infty)$ and $L_2[0, \infty)$.
- Invertible on $L_2(-\infty, \infty)$ but not on $L_2[0, \infty)$.

The shift operator can be defined by a multiplication operator

$$S_\tau = \phi^{-1} M_{\hat{S}} \phi$$

where

$$\hat{S}(\imath\omega) = e^{-\imath\omega t}$$

# Time-Invariant Systems

**Definition 4.**

An operator $Q$ is **Time-Invariant** if

$$S_\tau Q = Q S_\tau$$

for all $\tau > 0$.

- $(Qu)(t-\tau) = Q(S_\tau u)(t)$
- Initial time doesn't matter.
  - Identical signals applied at different times will produce the same output.
- Shifting the input shifts the output.

# Time-Invariant Systems

## Most Systems are Time-Invariant

Any state-space system is time-invariant

$$\dot{x}(t) = Ax(t) + Bu(t)$$

- Unless of course $A$ varies with time ($A(t)$)

**Lemma 5.**

*For any $\hat{G}$, $G = \phi^{-1} M_{\hat{G}} \phi$ is a time-invariant operator.*

Proof.

Recall a system is time-invariant if $S_\tau G = G S_\tau$. Examine the first term

$$\begin{aligned} S_\tau G &= \phi^{-1} M_{\hat{S}} \phi \phi^{-1} M_{\hat{G}} \phi \\ &= \phi^{-1} M_{\hat{S}} M_{\hat{G}} \phi \\ &= \phi^{-1} M_{\hat{S}\hat{G}} \phi \\ &= \phi^{-1} M_{\hat{G}\hat{S}} \phi \\ &= \phi^{-1} M_{\hat{S}} M_{\hat{S}} \phi \\ &= \phi^{-1} M_{\hat{G}} \phi \phi^{-1} M_{\hat{S}} \phi \\ &= G S_\tau \end{aligned}$$

This works because $\hat{S} = e^{-\imath\omega\tau} I$ and so $(\hat{G}\hat{S} = \hat{S}\hat{G})$. □

More significantly, the converse is also true.

**Theorem 6.**

*An operator* $G : L_2(-\infty,\infty) \to L_2(-\infty,\infty)$ *is time-invariant if and only if there exists some* $\hat{G} \in \hat{L}_\infty$ *such that*

$$G = \phi^{-1} M_{\hat{G}} \phi$$

- All LTI systems can be represented using transfer functions in $\hat{L}_\infty$.
- Note that not all transfer functions have a state-space representation. (e.g. delay)

Linear, Causal, Time-Invariant Systems are those which are well-defined on $L_2[0,\infty)$.

**Definition 7.**

Define the **Truncation Operator** $P_\tau : L_2(-\infty,\infty) \to L_2(-\infty,\infty)$ as

$$(P_\tau u)(t) = \begin{cases} u(t) & t \le \tau \\ 0 & t > \tau \end{cases}$$

Truncation operator zeros out the signal after time $\tau$.

# Causal Systems

An operator is causal if changes in the future input don't create changes in past output.

- If the output at time t, $y(t)$ only depends on the input up to time $t$, $u(s)$, $s \in (-\infty, t]$.

**Definition 8.**

An operator $G \in \mathcal{L}(L_2)$ is **Causal** if

$$P_\tau G P_\tau = P_\tau G$$

for all $\tau \in \mathbb{R}$.

# Causal Systems

**Lemma 9.**

*A linear time-invariant operator, $G$, is causal if and only if*

$$P_0 G P_0 = P_0 G$$

Proof.

First note that on $L_2(-\infty,\infty)$, $S_\tau$ is an invertible operator. Hence $P_\tau = S_\tau P_0 S_{-\tau}$: we can shift truncation point to $0$, truncate, then shift back.

- This implies $P_\tau S_\tau = S_\tau P_0$ (truncation is not a time-invariant operator).
- We have the following equivalence: $G$ is causal if and only if

| | |
|---|---|
| $\Leftrightarrow P_\tau G P_\tau = P_\tau G$ | |
| $\Leftrightarrow P_\tau G P_\tau S_\tau = P_\tau G S_\tau$ | $S_\tau$ is invertible |
| $\Leftrightarrow P_\tau G S_\tau P_0 = P_\tau S_\tau G$ | $G$ is LTI |
| $\Leftrightarrow P_\tau S_\tau G P_0 = S_\tau P_0 G$ | $G$ is LTI |
| $\Leftrightarrow S_\tau P_0 G P_0 = S_\tau P_0 G$ | $P_\tau S_\tau = S_\tau P_0$ |
| $\Leftrightarrow P_0 G P_0 = P_0 G$ | $S_\tau$ is invertible |

# Causal Systems

**Corollary 10.**

*If $G \in \mathcal{L}(L_2(-\infty,\infty))$ is LTI, then $G$ is causal if and only if*

$$G : L_2[0,\infty) \to L_2[0,\infty)$$

Thus the subspace of Linear Causal Time-Invariant Operators is $\mathcal{L}(L_2[0,\infty))$

Proof.

We first show that $2) \Rightarrow 1)$. Suppose $G : L_2[0,\infty) \to L_2[0,\infty)$.

- For any $u \in L_2(-\infty,\infty)$, $P_0 u \in L_2(-\infty,0] = L_2[0,\infty)^\perp$.
- Thus $(I - P_0)u \in L_2[0,\infty)$.
- Thus $G(I - P_0)u \in L_2[0,\infty)$.
- Thus $P_0 G(I - P_0)u = 0$.
- Thus $P_0 G = P_0 G P_0$.
- Hence $G$ is causal.

□

# Causal Systems

**Corollary 11.**

*If $G \in \mathcal{L}(L_2(-\infty,\infty))$ is LTI, then $G$ is causal if and only if*

$$G : L_2[0,\infty) \to L_2[0,\infty)$$

Proof.

Now we show that $1)$ implies $2)$. Suppose that $P_0 G = P_0 G P_0$. Then $P_0 G(I - P_0)u = 0$.

- Then $(I - P_0)G(I - P_0) = G(I - P_0)$.
- Note that for $u \in L_2[0,\infty)$, we have $(I - P_0)u = u$.
- Thus for $u \in L_2[0,\infty)$,

$$\begin{aligned} Gu &= G(I - P_0)u \\ &= (I - P_0)G(I - P_0)u \\ &\in L_2[0,\infty) \end{aligned}$$

since $(I - P_0)$ is the projection onto $L_2[0,\infty)$

# Summary

An LTI operator is causal iff it maps $L_2[0,\infty) \to L_2[0,\infty)$

- Any LTI operator is defined by a multiplication operator (transfer function).
- Which multiplication operators map $H_2 \to H_2$ (causal operators)
  - Which operators define causal systems?

**Next Lecture:** $H_\infty$.