Modern Control Systems

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Lecture 16: H_∞ and Summary of Linear Analysis

So far we know:

- The Fourier Transform, ϕ maps $L_2(-\infty,\infty)$ to $\hat{L}_2$.
- The Laplace Transform, Λ maps $L_2[0,\infty)$ to H_2.
- A Transfer Function is any element $\hat{G} \in \hat{L}_\infty$.
- A Transfer function defines a multiplication operator $M_{\hat{G}}$ which maps $\hat{L}_2$ to $\hat{L}_2$.
- Any Linear, Time-Invariant System $G : L_2 \to L_2$ can be represented by a transfer function as $\phi^{-1} M_{\hat{G}} \phi$ for some $\hat{G} \in \hat{L}_\infty$.

Question: How do we represent *Causal* Systems, which map $H_2 \to H_2$?

The Space H_∞

Definition 1.

A function $\hat{G} : \bar{\mathbb{C}}^+ \to \mathbb{C}^{n \times m}$ is in H_∞ if

1. $\hat{G}(s)$ is analytic on the CRHP, $\mathbb{C}^+$.
2. $$\lim_{\sigma \to 0^+} \hat{G}(\sigma + \imath\omega) = \hat{G}(\imath\omega)$$
3. $$\sup_{s \in \mathbb{C}^+} \bar{\sigma}(\hat{G}(s)) < \infty$$

- Similar to $\hat{L}_\infty$, but analytic.
- Elements of $\hat{L}_\infty$ with an analytic continuation to the right half-plane.
- A Banach Space with norm

$$\|\hat{G}\|_{H_\infty} = \operatorname{ess}\sup_{\omega \in \mathbb{R}} \bar{\sigma}(\hat{G}(\imath\omega))$$

The Space H_∞

For any analytic functions, $\hat{u}$ and $\hat{G}$, the function

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

is analytic. Thus if $\hat{G} \in H_\infty$,

- $\hat{G}$ is analytic on CRHP
- $M_{\hat{G}} : H_2 \to H_2$.
- $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ maps $L_2[0,\infty) \to L_2[0,\infty)$.
- $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ is causal, LTI.

Causal Systems and H_∞

Indeed, this is necessary and sufficient.

Theorem 2.

G is a Causal, Linear, Time-Invariant Operator on L_2 if and only if there exists some $\hat{G} \in H_\infty$ such that $G = \Lambda^{-1} M_{\hat{G}} \Lambda$.

$$(\Lambda G u)(\imath\omega) = \hat{G}(\imath\omega)\hat{u}(\imath\omega)$$

Conclusion: H_∞ provides a complete parameterization of the Banach space of causal bounded linear time-invariant operators with

$$\|G\|_{\mathcal{L}(L_2[0,\infty))} = \|\Lambda^{-1} M_{\hat{G}} \Lambda\|_{\mathcal{L}(L_2[0,\infty))} = \|\hat{G}\|_{H_\infty}$$

Optimal Control is an attempt to minimize the H_∞ norm of the closed-loop transfer function.

Example of H_∞

Example:

$$\hat{G}(\imath\omega) = \frac{e^{-\imath\omega\tau} - 1}{\imath\omega}$$

which has

$$\|\hat{G}\|_{H_\infty} = \tau$$

which defines the system

$$y(t) = \int_0^t \left(u(s-\tau) - u(s)\right) ds$$

Question: How to parameterize H_∞?

Rational Transfer Functions

The space of bounded analytic functions, H_∞, is infinite-dimensional.

- this makes it hard to design optimal controllers.

We often restrict ourselves to state-space systems and state-space controllers.

Definition 3.

The space of rational functions is defined as

$$R := \left\{ \frac{p(s)}{q(s)} \; : \; p, q \text{ are polynomials} \right\}$$

We define the following rational subspaces.

$$RH_2 = R \cap H_2$$
$$R\hat{L}_2 = R \cap \hat{L}_2$$
$$RH_\infty = R \cap H_\infty$$

Note that RH_2, $R\hat{L}_2$ and RH_∞ are not complete spaces.

Rational Transfer Functions

For rational transfer functions, the set of bounded LTI systems are precisely those with no unstable poles.

Definition 4.

- A rational function $r(s) = \frac{p(s)}{q(s)}$ is **Proper** if the degree of p is less than or equal to the degree of q.
- A rational function $r(s) = \frac{p(s)}{q(s)}$ is **Strictly Proper** if the degree of p is less than the degree of q.

Proposition 1.

1. $\hat{G} \in R\hat{L}_\infty$ *if and only if* $\hat{G}$ *is proper with no poles (roots of* $q(s)$*) on the imaginary axis.*
2. $\hat{G} \in RH_\infty$ *if and only if* $\hat{G}$ *is proper with no poles on the closed right half-plane.*

State-Space Systems

Recall a State-space

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Theorem 5.

- *For any stable state-space system, G, there exists some $\hat{G} \in RH_\infty$ such that*

$$G = \Lambda^{-1} M_{\hat{G}} \Lambda$$

- *For any $\hat{G} \in RH_\infty$, the operator $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ can be represented in state-space for some A, B, C and D where A is Hurwitz.*

For state-space system, (A, B, C, D),

$$\hat{G}(s) = C(sI - A)^{-1} B + D$$

State-Space is NOT Unique

- For a given Causal LTI system G with transfer function, $\hat{G} \in RH_\infty$, there may be many state-space representations

Equivalent Realizations

Definition 6.

Two state-space representations, (A, B, C, D) and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ are **Equivalent** if

$$C(sI - A)^{-1}B + D = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$$

Definition 7.

A representation, (A, B, C, D) is **Minimal** if it is controllable and observable.

Lemma 8.

Any transfer function $\hat{G} \in RH_\infty$ has a minimal state-space representation.

We are skipping the section on minimality.

- We will, however, return to the question of Grammians.