Grammians

Matthew M. Peet
Arizona State University

Lecture 17: Grammians

Lyapunov Equations

Proposition 1.

Suppose A is Hurwitz and Q is a square matrix. Then

$$X = \int_0^\infty e^{A^T s} Q e^{As} ds$$

is the unique solution to the Lyapunov Equation

$$A^T X + XA + Q = 0$$

Proposition 2.

Suppose $Q > 0$. Then A is Hurwitz is and only if there exists a solution $X > 0$ to the Lyapunov equation

$$A^T X + XA + Q = 0$$

Lyapunov Inequalities

Proposition 3.

Suppose A is Hurwitz and $X_1 \geq 0$ satisfies

$$A^T X_1 + X_1 A = -Q$$

Suppose X_2 satisfies

$$A^T X_2 + X_2 A < -Q.$$

Then $X_2 > X_1$.

Proof.

$$\begin{aligned} A^T(X_2 - X_1) + (X_2 - X_1)A &= (A^T X_2 + X_2 A) - (A^T X_1 + X_1 A) \\ &= A^T X_2 + X_2 A + Q < 0 \end{aligned}$$

Since A is Hurwitz and $Q > 0$, by the previous Proposition $X_2 - X_1 > 0$

□

Grammians

Recall From State-Space Systems:

- Controllable means we can do eigenvalue assignment.
- Observable means we can design an observer.
- Controllable and Observable means we can design an observer-based controller.

Questions:

- How difficult is the control problem?
- What is the effect of an input on an output?

Grammians

To give quantitative answers to these questions, we use Grammians.

Definition 1.

For pair (C, A), the **Observability Grammian** is defined as

$$Y = \int_0^\infty e^{A^T s} C^T C e^{As} ds$$

Definition 2.

The **Controllability Grammian** of pair (A, B) is

$$W := \int_0^\infty e^{As} B B^T e^{A^T s} ds$$

Grammians

Grammians are linked to Observability and Controllability

Theorem 3.

For a given pair (C, A)*, the following are equivalent.*

- $\ker Y = 0$
- $\ker \Psi_o = 0$
- $\ker O(C, A) = 0$

Theorem 4.

For any $t \geq 0$*,*

$$R_t = C_{AB} = \textit{Image}\ (W_t)$$

Observability Grammian

Recall the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Assume that A is Hurwitz.
Recall the Observability Operator $\Psi_o : \mathbb{R}^n \to L_2[0, \infty)$.

$$(\Psi_o x_0)(t) = \begin{cases} Ce^{At}x_0 & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

- $\Psi_o x_0 \in L_2$ because A is Hurwitz.
- When $u = 0$, this is also the solution.
- We would like to look at the "size" of the output produced by an initial condition.
    - Now we know how to measure the "size" of the output signal.

$$\|y\|_{L_2}^2 = \langle \Psi_o x_0, \Psi_o x_0 \rangle_{L_2} = \langle x_0, \Psi_o^* \Psi_o x_0 \rangle_{\mathbb{R}^n}$$

- How to calculate the adjoint $\Psi_o^* : L_2 \to \mathbb{R}^n$?

Observability Grammian

It can be easily confirmed that the adjoint of the observability operator is

$$\Psi_o^* z = \int_0^\infty e^{A^T s} C^T z(s) ds$$

Then

$$\Psi_o^* \Psi_o x_0 = \left[\int_0^\infty e^{A^T s} C^T C e^{As} ds \right] x_0$$

Which is simply the observability grammian

$$Y_o = \Psi_o^* \Psi_o = \int_0^\infty e^{A^T s} C^T C e^{As} ds$$

Recall from the HW: Y_o is the solution to

$$A^* Y_o + Y_o A + C^T C = 0$$

and $Y_o > 0$ if and only if (C, A) is observable.

Observability Grammian

Proposition 4.

Then (C, A) *is observable if only if there exists a solution* $X > 0$ *to the Lyapunov equation*

$$A^T X + XA + C^T C = 0$$

The physical interpretation is clear: how much does an initial condition affect the output in the L_2-norm

$$\|y\|_{L_2} = x_0^T Y_o x_0$$

Since this is just a matrix, we can take this further by looking at which directions are most observable.

- Will correspond to $\bar{\sigma}(Y_o)$.

Definition 5.

The **Observability Ellipse** is

$$E_o := \left\{ x : x = Y_o^{1/2} x_0,\ \|x_0\| = 1 \right\}$$

Definition 6.

The **Observability Ellipse** is

$$E_o := \left\{ x : x = Y_o^{1/2} x_0,\ \|x_0\| = 1 \right\}$$

Notes:

1. E_o is an ellipse.
$$E_o = \{x :\ x^T Y_o^{-1} x = 1\}$$
For a proof,
   - let $x \in E_o$. Then there exists some $|x_0| = 1$ such that $x_0 = Y_o^{-1/2} x$.
   - Then $x^T Y_o^{-1} x = x^T Y_o^{-1/2} Y_o^{-1/2} x = |x_0|^2 = 1$.
   - Thus $E_o \subset \{x :\ x^T Y_o^{-1} x = 1\}$. The other direction is similar
2. The Principal Axes of E_0 are the eigenvectors of $Y_o^{1/2}$, u_i.
3. The lengths of the Principal Axes of E_0 are $\sigma_i(Y_o)$.
4. If $\sigma_i(Y_o) = 0$, the u_i is in the unobservable subspace.

Controllability Operator

Recall the Controllability Operator $\Psi_c : L_2(-\infty, 0] \to \mathbb{C}^n$

$$\Psi_c u = \int_{-\infty}^{0} e^{-As} B u(s) ds$$

Which maps an input to a final state $x(0)$.

- Adjoint $\Psi_c^* : \mathbb{R}^n \to L_2(-\infty, 0]$

$$\left(\Psi_c^* x\right)(t) = B^* e^{-A^* t} x$$

Recall: The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if for any $x(0) \in \mathbb{R}^n$, there exists some $u \in L_2(-\infty, 0]$ such that

$$x(0) = \Psi_c u$$

Controllability Grammian

Definition

Definition 7.

The Controllability Grammian is

$$X_c := \Psi_c \Psi_c^* = \int_{-\infty}^{0} e^{-As} B B^T e^{-A^T s} ds$$
$$= \int_{0}^{\infty} e^{As} B B^T e^{A^T s} ds$$

Recall

- X_c is the solution to

$$A X_c + X_c A^T + B B^T = 0$$

- $X_c > 0$ if and only if (A, B) is controllable.

Observability Grammian

Proposition 5.

Then (A, B) *is controllable if only if there exists a solution* $X > 0$ *to the Lyapunov equation*

$$AX + XA^T + BB^T = 0$$

Controllability Grammian

Proposition 6.

Suppose (A, B) *is controllable. Then*

1. X_c *is invertible*
2. *Given* x_0, *the solution to*

$$\min_{u \in L_2(-\infty,0]} \|u\|_{L_2}:$$
$$x_0 = \Psi_c u$$

is given by

$$u_{opt} = \Psi_c^* X_c^{-1} x_0$$

Controllability Grammian

Proof.

The first is clear from $X_c > 0$. For the second part, we first show that u_{opt} is feasible. We then show that it is optimal.

- For feasibility, we note that

$$\begin{aligned}\Psi_c u_{opt} &= \Psi_c \Psi_c^* X_c^{-1} x_0 \\ &= X_c X_c^{-1} x_0 \\ &= x_0\end{aligned}$$

  which implies feasibility

□

Controllability Grammian

Proof.

Now that we know that u_{opt} is feasible, we show that for any other $\bar{u}$, if $\bar{u}$ is feasible, then $\|\bar{u}\|_{L_2} \ge \|u_{opt}\|_{L_2}$.

- Define $P := \Psi_c^* X_c^{-1} \Psi_c$.

$$P^2 = \Psi_c^* X_c^{-1} \underbrace{\Psi_c \Psi_c^*}_{X_c} X_c^{-1} \Psi_c = \Psi_c^* X_c^{-1} \Psi_c = P$$

- Furthermore $P^* = P$.
- Thus P is a projection operator, which means

$$\langle Pu, (I-P)u \rangle = 0$$

- Thus for any $\bar{u}$

$$\|\bar{u}\|^2 = \|P\bar{u} + (I-P)\bar{u}\|^2 = \|P\bar{u}\|^2 + \|(I-P)\bar{u}\|^2.$$

□

Controllability Grammian

Proof.

- If $\bar{u}$ is feasible, then

$$\begin{aligned}\|P\bar{u}\|^2 &= \|\Psi_c^* X_c^{-1} \Psi_c \bar{u}\| \\ &= \|\Psi_c^* X_c^{-1} x_0\| \qquad \text{since } \bar{u} \text{ is feasible} \\ &= \|u_{opt}\|^2\end{aligned}$$

- We conclude that

$$\|\bar{u}\|^2 = \|u_{opt}\|^2 + \|(I-P)\bar{u}\|^2 \geq \|u_{opt}\|^2$$

- Thus u_{opt} is optimal

□

This shows that u_{opt} is the minimum-energy input to achieve the final-state x_0.
Drawbacks:

- Don't have infinite time.
- Open-loop

Controllability Grammian

Physical Interpretation

The controllability Grammian tells us the minimum amount of energy required to reach a state.

$$\|u_{opt}\|_{L_2}^2 = x_0^T X_c^{-1} x_0$$

Definition 8.

The **Controllability Ellipse** is the set of states which are reachable with 1 unit of energy.

$$\{\Psi_c u \,:\, \|u\|_{L_2} \le 1\}$$

Proposition 7.

The following are equivalent

1. $\{\Psi_c u \,:\, \|u\|_{L_2} \le 1\}$
2. $\left\{X_c^{1/2} x \,:\, \|x\| \le 1\right\}$
3. $\left\{x \,:\, x^T X_c^{-1} x \le 1\right\}$

Because we don't always have infinite time:

- What is the optimal way to get to x in time T

Finite-Time Controllability Operator: $\Psi_T : L_2[0,T] \to \mathbb{R}^n$.

$$\Psi_T u := \int_0^T e^{A(T-s)} B u(s) ds$$

Finite-Time Controllability Grammian

$$X_T := \Psi_T \Psi_T^* = \int_0^T e^{As} B B^T e^{A^T s} ds$$

Note: $X_T \geq X_s$ for $t \geq s$.

Controllability Grammian

Finite-Time Grammian

Cannot be found by solving the Lyapunov equation.
Must be found by numerical integration of the matrix-differential equation:

$$\dot{X}_T(t) = AX_T(t) + X_T(t)A^T + BB^T$$

from $t = 0$ to $t = T$ with $X_T(0) = 0$.

- X_c is the steady-state solution.

Proposition 8.

Suppose (A, B) is controllable. Then

1. X_T *is invertible*
2. *The solution to*

$$\min_{u \in L_2[0,T]} \|u\|_{L_2}:$$

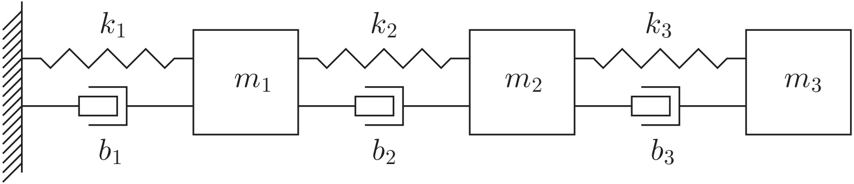
$$x_f = \Psi_T u$$

is given by

$$u_{opt} = \Psi_T^* X_T^{-1} x_f$$

Finite-Time Grammian

Example



Consider the Spring-mass system ($k_i = m_i = 1$, $b_i = .8$)

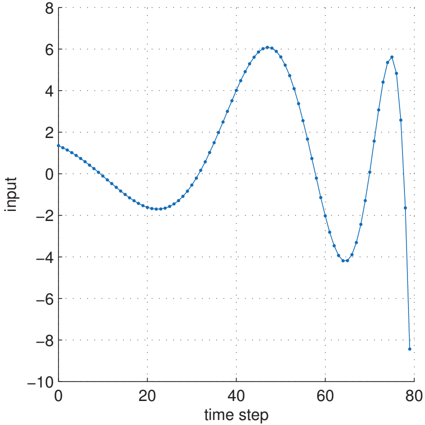
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & .8 & 0 \\ 1 & -2 & 1 & .8 & -1.6 & .8 \\ 0 & 1 & -1 & 0 & .9 & -.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

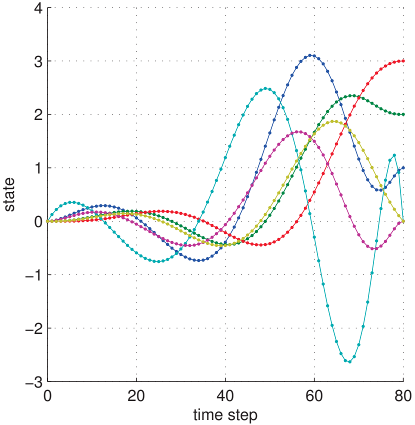
with desired final state

$$x_f = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$$

Finite-Time Grammian

Example





$$x_f = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$$