2-input 2-output Framework

We introduce the control framework by separating internal signals from external signals.

Output Signals:

- **z:** Output to be controlled/minimized
  - Regulated output
- **y:** Output used by the controller
  - Must be measured in real-time by sensor
  - May replicate signals from regulated output

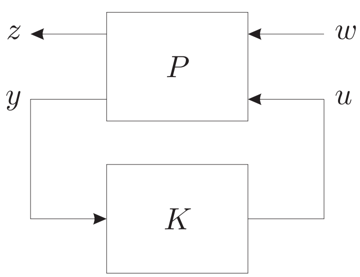
2-input 2-output Framework

Input Signals:

- **w:** Disturbance, Tracking Signal, etc.
  - exogenous input
- **u:** Output from controller
  - Input to actuator
  - Not related to external input

The Optimal Control Framework

The controller closes the loop from y to u.



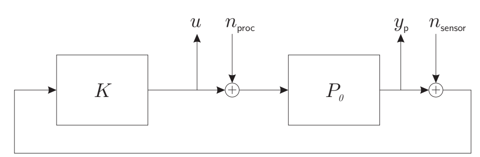
For a linear system P, we have 4 subsystems.

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

All G_{ij} are MIMO

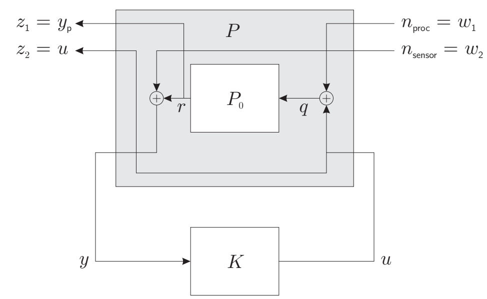
$$P_{11} : w \mapsto z \qquad P_{12} : u \mapsto z$$
$$P_{21} : w \mapsto y \qquad P_{22} : u \mapsto y$$

The Regulator

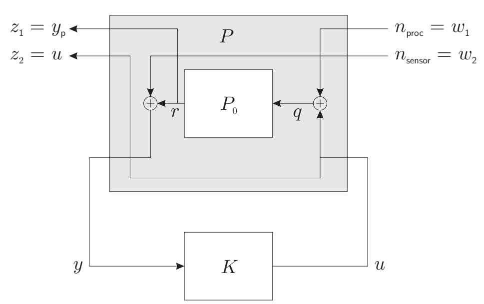


If we define

$$z_2 = u \qquad q = w_1 + w_2$$
$$z_1 = y_p \qquad y = r + w_2$$



The Regulator



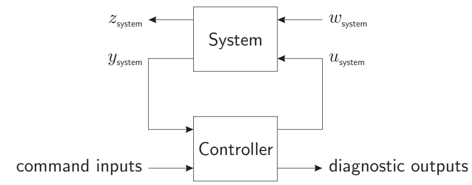
The reconfigured plant P is given by

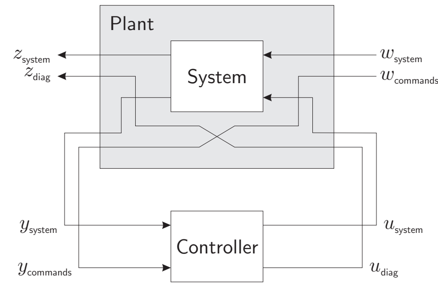
$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} P_0 & 0 & P_0 \\ 0 & 0 & I \\ P_0 & I & P_0 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \\ u(t) \end{bmatrix}$$

If $P_0 = (A, B, C, D)$, then

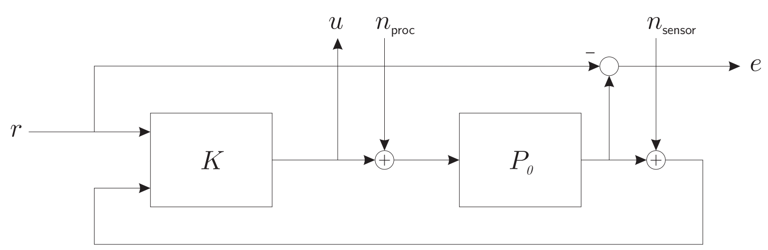
$$P = \left[\begin{array}{c|ccc} A & B & 0 & B \\ \hline C & D & 0 & D \\ 0 & 0 & 0 & I \\ C & D & I & D \end{array}\right]$$

Diagnostics



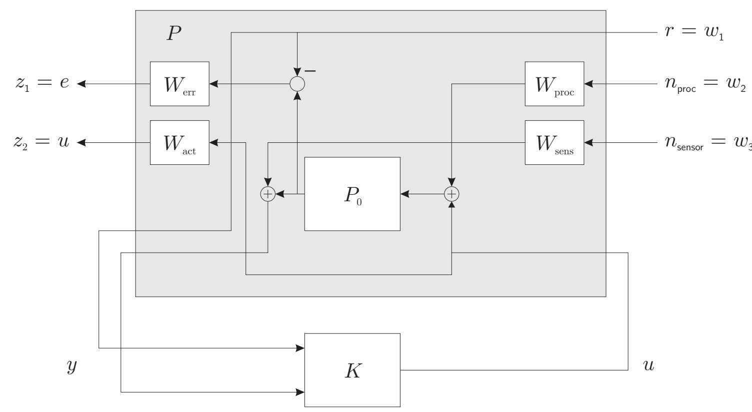


Tracking Control



$$\begin{aligned} r &= \text{tracking input} & w_2 &= n_{proc} & w_1 &= r \\ e &= \text{tracking error} & w_3 &= n_{sensor} & u &= u \\ n_{proc} &= \text{process noise} & z_1 &= e & y_1 &= r \\ n_{sensor} &= \text{sensor noise} & z_2 &= u & y_2 &= y_p \end{aligned}$$

Tracking Control

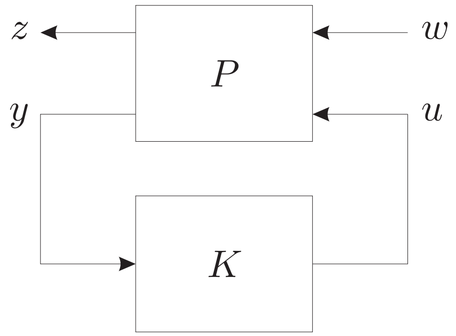


$$P = \begin{bmatrix} I & -P_0 & 0 & -P_0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & P_0 & I & P_0 \end{bmatrix}$$

$$\begin{aligned} z_1 &= r - P_0(n_{proc} + u) \\ z_2 &= u \\ y_1 &= r \\ y_2 &= w_3 + P_0(n_{proc} + u) \end{aligned}$$

Linear Fractional Transformation

Close the loop



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Linear Fractional Transformation

$$\begin{aligned} z &= P_{11}w + P_{12}u \\ y &= P_{21}w + P_{22}u \\ u &= Ky \end{aligned}$$

Solving for u,

$$u = KP_{21}w + KP_{22}u$$

Thus

$$\begin{aligned} (I - KP_{22})u &= KP_{21}w \\ u &= (I - KP_{22})^{-1}KP_{21}w \end{aligned}$$

Now we solve for z:

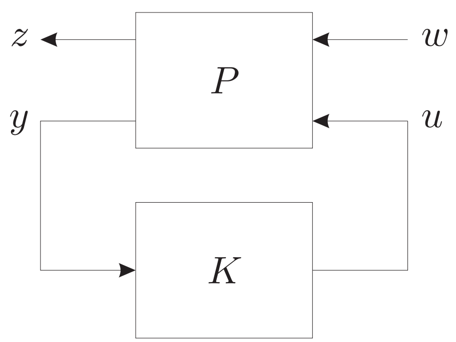
$$z = \left[P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\right] w$$

Linear Fractional Transformation

This expression is called the Linear Fractional Transformation of (P, K), denoted

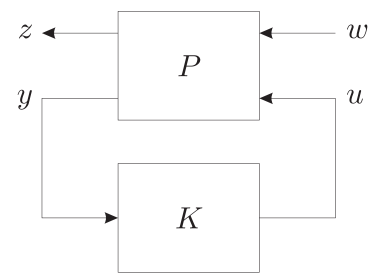
$$\underline{S}(P, K) := P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}$$

AKA: Lower Star Product



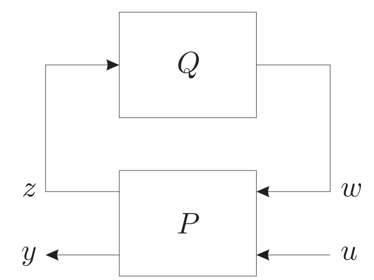
Other Fractional Transformations

Lower LFT:



$$\underline{S}(P, K) := P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}$$

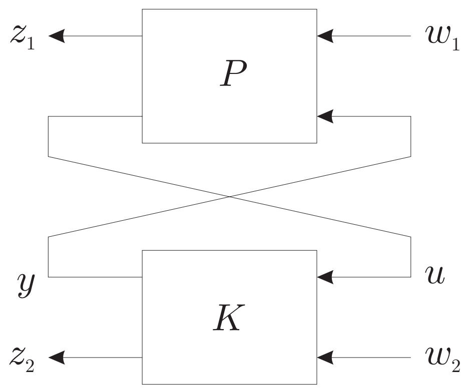
Upper LFT:



$$\bar{S}(P, K) := P_{22} + P_{21}Q(I - P_{11}Q)^{-1}P_{12}$$

Other Fractional Transformations

Star Product:



$$S(P,K) := \begin{bmatrix} \underline{S}(P,K_{11}) & P_{12}(I-K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I-P_{22}K_{11})^{-1}P_{21} & \bar{S}(K,P_{22}) \end{bmatrix}$$

Well-Posedness

The interconnection doesn't always make sense.

Definition 1.

The interconnection $\underline{S}(P,K)$ is **well-posed** if for any smooth w and any $x(0)$ and $x_K(0)$, there exist functions x, x_K, u, y, z such that

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) & \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) & u(t) &= C_K x_K(t) + D_K y(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{aligned}$$

Note: The solution does not need to be in L_2.

- Says nothing about stability.

Well-Posedness

In state-space format:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_K(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t)$$

$$z(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + D_{11} w(t)$$

From

$$\begin{aligned} u(t) &= D_K y(t) + C_K x_K(t) \\ y(t) &= D_{22} u(t) + C_2 x(t) + D_{21} w(t) \end{aligned}$$

We have

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$

Because the rest is state-space, the interconnection is well-posed if and only if the matrix $\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}$ is invertible.

Well-Posedness

Question: When is

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}$$

invertible?

Answer: 2x2 matrices have a closed-form inverse

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix}$$

where $Q = (I - D_{22} D_K)^{-1}$.

Proposition 1.

The interconnection $\underline{S}(P,K)$ is well-posed if and only if $(I - D_{22}D_K)$ is invertible.

- Equivalently $(I - D_K D_{22})$ is invertible.
- Sufficient conditions: $D_K = 0$ or $D_{22} = 0$.
- To optimize over K, we will need to enforce this constraint somehow.

Optimal Control

Definition 2.

The **Optimal H_∞-Control Problem** is

$$\min_{K \in H_\infty} \|\underline{S}(P,K)\|_{H_\infty} = \|\underline{S}(P,K)\|_{\mathcal{L}(L_2)}$$

- Also Optimal H_∞ dynamic-output-feedback Control Problem

Definition 3.

The **Optimal H_2-Control Problem** is

$$\min_{K \in H_\infty} \|\underline{S}(P,K)\|_{H_2} \quad \text{such that}$$

$$\underline{S}(P,K) \in H_\infty.$$

Optimal Control

Choose K to minimize

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|_{H_\infty}$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right] \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

In either case, the problem is Nonlinear.

Optimal Control

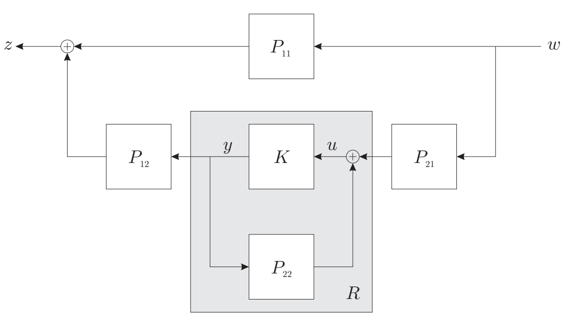
There are several ways to address the problem of nonlinearity.

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|_{H_\infty}$$

Variable Substitution: The easiest way to make the problem linear is by declaring a new variable $\quad R := (I - KP_{22})^{-1}K$

The optimization problem becomes: Choose R to minimize

$$\|P_{11} + P_{12}RP_{21}\|_{H_\infty}$$



Optimal Control

We optimize

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|_{H_\infty} = \|P_{11} + P_{12}RP_{21}\|_{H_\infty}$$

Once, we have the optimal R, we can recover the optimal K as

$$K = R(I + RP_{22})^{-1}$$

Problems:

- how to optimize $\|\cdot\|_{H_\infty}$.
- Is the controller stable?
  - Does the inverse $(I + RP_{22})^{-1}$ exist? Yes.
  - Is it a bounded linear operator?
  - In which space?
- An important branch of control.
  - Coprime factorization
  - Youla parameterization
- We will sidestep this body of work.