Modern Optimal Control

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Lecture 19: Stabilization via LMIs

Optimization

Optimization can be posed in functional form:

$$\min_{x \in \mathbb{F}} \; \texttt{objective function}: \qquad \text{subject to}$$
$$\texttt{inequality constraints}$$

which may have the form

$$\min_{x \in \mathbb{F}} f_0(x): \qquad \text{subject to}$$
$$f_i(x) \ge 0 \qquad i = 1, \cdots k$$

Special Cases:

- **Linear Programming**
  - If the $f_i(x) : \mathbb{R}^n \to \mathbb{R}^m$ are affine ($f_i(x) = Ax - b$)
  - EASY: Simplex/Ellipsoid Algorithm
- **Polynomial Programming**
  - The $f_i(x) : \mathbb{R}^n \to \mathbb{R}^m$ are polynomial.
- **Semidefinite Programming**
  - The $f_i(x) : \mathbb{R}^n \to \mathbb{S}^m$ are affine.

For semidefinite programming, what does $f_i(x) \ge 0$ mean?

Positive Matrices

Definition 1.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Definition 2.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \qquad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

Positive Matrices

Lemma 3.

$P \in \mathbb{S}^n$ is positive definite if and only if all its eigenvalues are positive.

Easy Proofs:

- A Positive Definite matrix is invertible.
- The inverse of a positive definite matrix is positive definite.
- If $P > 0$, then $TPT^T \geq 0$ for any T. If T is invertible, then $TPT^T > 0$.

Lemma 4.

For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ such that $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

Convex Optimization

Why is Linear Programming easy and polynomial programming hard?

$$\min_{x\in\mathbb{F}} f_0(x) : \qquad \text{subject to}$$
$$f_i(x) \ge 0 \qquad i = 1, \cdots k$$

Consider the **geometric representation** of the optimization problem:

$$\min_{x\in\mathbb{F}} \; f_0(x) : \qquad \text{subject to}$$
$$x \in S$$

where $S := \{x \,:\, f_i(x) \ge 0,\; i = 1, \cdots, k\}$.

or the more compact *pure* geometric representation

$$\min_{\gamma, x\in\mathbb{F}} \; \gamma : \qquad \text{subject to}$$
$$(\gamma, x) \in S'$$

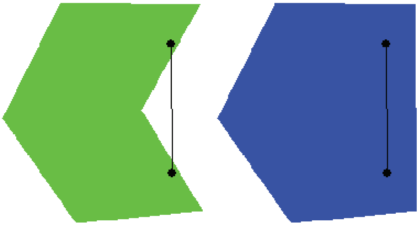
where $S' := \{(\gamma, x) \,:\, \gamma - f_0(x) \ge 0,\; f_i(x) \ge 0,\; i = 1, \cdots, k\}$.

Definition 5.

A set is **convex** if for any $x, y \in Q$,

$$\{\mu x + (1-\mu)y \,:\, \mu \in [0,1]\} \subset Q.$$

The line connecting any two points lies in the set.



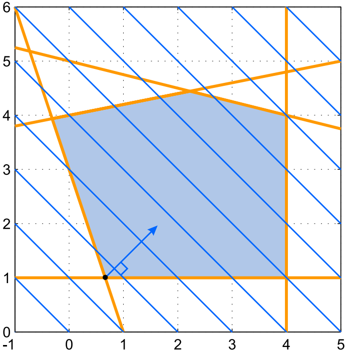
Convex Optimization:

Definition 6.

Consider the optimization problem

$$\min_{\gamma, x \in \mathbb{F}} \; \gamma : \qquad \text{subject to}$$
$$(\gamma, x) \in S'.$$

The problem is **Convex Optimization** if the set S' is convex.



Convex optimization problems have the property that scaled Newton iteration always converges to the global optimal.

The question is, of course, when is the set S' convex?

- For polynomial optimization, a sufficient condition is that all functions f_i are convex.
  - **The set of convex functions is a convex cone.**
  - The level set of a convex function is a convex set.

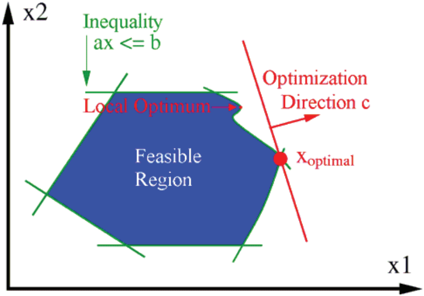
Convexity

Newton's Algorithm: Solve $f(x) = 0$

$$x_{k+1} = x_k - t\frac{f(x_k)}{f'(x_k)}$$

t is the step-size.

For non-convex optimization, Newton descent may get stuck at local optima.



For Newton descent, constraints are represented by barrier functions.

Semidefinite Programming

$$\begin{array}{lll} \text{minimize} & \operatorname{trace} CX & \\ \text{subject to} & \operatorname{trace} A_i X = b_i & \text{for all } i \\ & X \succeq 0 & \end{array}$$

- The variable X is a symmetric matrix
- $X \succeq 0$ means X is positive semidefinite
- The feasible set is the intersection of an affine set with the *positive semidefinite cone*
$$\{ X \in \mathbb{S}^n \mid X \succeq 0 \}$$

SDPs with Explicit Variables

We can also explicitly parameterize the affine set to give

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *linear matrix inequality (LMI)*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Linear Matrix Inequalities

Review

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.

Semidefinite Programming

$$\min c^T x :$$

$$Y_0 + \sum_i x_i Y_i > 0$$

Here the Y_i are symmetric matrices.

Linear Matrix Inequalities

$$\text{Find} x :$$

$$Y_0 + \sum_i x_i Y_i > 0$$

Commonly Takes the Form

$$\text{Find} X :$$

$$\sum_i A_i X B_i + Q > 0$$

Common Form:

$$\text{Find} X :$$
$$\sum_i A_i X B_i + Q > 0$$

The most important Linear Matrix Inequality is the *Lyapunov Inequality*.

There are several very efficient **LMI/SDP Solvers** for Matlab:

- **SeDuMi**
  - Fast, but somewhat unreliable.
  - See `http://sedumi.ie.lehigh.edu/`
- **LMI Lab** (Part of Matlab's Robust Control Toolbox)
  - Universally disliked
  - See `http://www.mathworks.com/help/robust/lmis.html`
- **YALMIP** (a parser for other solvers)
  - See `http://users.isy.liu.se/johanl/yalmip/`

I recommend YALMIP with solver SeDuMi.

Some Simple examples of LMI conditions in control include:

- Stability

$$A^T X + XA \prec 0$$
$$X \succ 0$$

- Stabilization

$$AX + BZ + XA^T + Z^T B^T \prec 0$$
$$X \succ 0$$

- H_2 Synthesis

$$\min Tr(W)$$
$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T \prec 0$$
$$\begin{bmatrix} X & (CX + DZ)^T \\ (CX + DZ) & W \end{bmatrix} \succ 0$$

We will go beyond these examples.

LMIs unite time-domain and frequency-domain analysis

Theorem 7 (Lyapunov).

Suppose there exists a continuously differentiable function V *for which* $V(0) = 0$ *and* $V(x) > 0$ *for* $x \neq 0$. *Furthermore, suppose* $\lim_{\|x\| \to \infty} V(x) = \infty$ *and*

$$\lim_{h \to 0^+} \frac{V(x(t+h)) - V(x(t))}{h} = \frac{d}{dt} V(x(t)) < 0$$

for any x *such that* $\dot{x}(t) = f(x(t))$. *Then for any* $x(0) \in \mathbb{R}$ *the system of equations*

$$\dot{x}(t) = f(x(t))$$

has a unique solution which is stable in the sense of Lyapunov.

The Lyapunov Inequality

Lemma 8.

A is Hurwitz if and only if there exists a $P > 0$ such that

$$A^T P + PA < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P + PA < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T A^T P x(t) + x(t)^T P A x(t) \\ &= x(t)^T \left(A^T P + PA\right) x(t) \end{aligned}$$

- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

The Lyapunov Inequality

Proof.

For the other direction, if A is Hurwitz, let

$$P = \int_0^\infty e^{A^T s} e^{As} ds$$

- Converges because A is Hurwitz.
- Furthermore
$$\begin{aligned} PA &= \int_0^\infty e^{A^T s} e^{As} A ds \\ &= \int_0^\infty e^{A^T s} A e^{As} ds = \int_0^\infty e^{A^T s} \frac{d}{ds}\left(e^{As}\right) ds \\ &= \left[e^{A^T s} e^{As} \right]_0^\infty - \int_0^\infty \frac{d}{ds} e^{A^T s} e^{-As} \\ &= -I - \int_0^\infty A^T e^{A^T s} e^{-As} = -I - A^T P \end{aligned}$$
- Thus $PA + A^T P = -I < 0$.

The Lyapunov Inequality

Other Versions:

Lemma 9.

(A, B) *is controllable if and only if there exists a* $X > 0$ *such that*

$$A^T X + XA + BB^T \leq 0$$

Lemma 10.

(C, A) *is stabilizable if and only if there exists a* $X > 0$ *such that*

$$AX + XA^T + C^T C \leq 0$$

These also yield Lyapunov functions for the system.

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 11.

The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= Kx(t)\end{aligned}$$

is stable

We have already solved the problem earlier using the Controllability Grammian.

- Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

Find $X > 0, K$:

$$X(A + BK) + (A + BK)^T X < 0$$

Problem: Bilinear in K and X.

The Static State-Feedback Problem

- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.
- To convexify the problem, we use a change of variables.

Problem 1:

$$\textbf{Find } X > 0, K \ :$$
$$X(A+BK)+(A+BK)^T X < 0$$

Problem 2:

$$\textbf{Find } P > 0, Z \ :$$
$$AP + BZ + PA^T + ZB^T < 0$$

Definition 12.

Two optimization problems are equivalent if a solution to one will provide a solution to the other.

Theorem 13.

Problem 1 is equivalent to Problem 2.

The Dual Lyapunov Equation

Problem 1:

Find $X > 0, :$

$XA + A^T X < 0$

Problem 2:

Find $Y > 0, :$

$YA^T + AY < 0$

Lemma 14.

Problem 1 is equivalent to problem 2.

Proof.

First we show 1) solves 2). Suppose $X > 0$ is a solution to Problem 1. Let $Y = X^{-1} > 0$.

- If $XA + A^T X < 0$, then

$$X^{-1}(XA + A^T X)X^{-1} < 0$$

- Hence

$$X^{-1}(XA + A^T X)X^{-1} = AX^{-1} + X^{-1}A^T = AY + YA^T < 0$$

- Therefore, Problem 2 is feasible with solution $Y = X^{-1}$.

The Dual Lyapunov Equation

Problem 1:

Find $X > 0, :$

$XA + A^T X < 0$

Problem 2:

Find $Y > 0, :$

$YA^T + AY < 0$

Proof.

Now we show 2) solves 1) in a similar manner. Suppose $Y > 0$ is a solution to Problem 1. Let $X = Y^{-1} > 0$.

- Then

$$\begin{aligned} XA + A^T X &= X(AX^{-1} + X^{-1}A^T)X \\ &= X(AY + YA^T)X < 0 \end{aligned}$$

□

Conclusion: If $V(x) = x^T P x$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^T P^{-1} x$ proves stability of $\dot{x} = A^T x$.

The Stabilization Problem

Thus we rephrase Problem 1

Problem 1:

Find $P > 0, K$:

$$(A + BK)P + P(A + BK)^T < 0$$

Problem 2:

Find $X > 0, Z$:

$$AX + BZ + XA^T + ZB^T < 0$$

Theorem 15.

Problem 1 is equivalent to Problem 2.

Proof.

We will show that 2) Solves 1). Suppose $X > 0$, Z solves 2). Let $P = X > 0$ and $K = ZP^{-1}$. Then $Z = KP$ and

$$\begin{aligned}(A + BK)P + P(A + BK)^T &= AP + PA^T + BKP + PK^TB^T \\ &= AP + PA^T + BZ + Z^TB^T < 0\end{aligned}$$

Now suppose that $P > 0$ and K solve 1). Let $X = P > 0$ and $Z = KP$. Then

$$AP + PA^T + BZ + Z^TB^T = (A + BK)P + P(A + BK)^T < 0$$

The Stabilization Problem

The result can be summarized more succinctly

Theorem 16.

(A, B) is static-state-feedback stabilizable if and only if there exists some $P > 0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0$$

with $u(t) = ZP^{-1}x(t)$.

Standard Format:

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} P \\ Z \end{bmatrix} + \begin{bmatrix} P & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix} < 0$$

There are a number of general-purpose LMI solvers available. e.g.

- `SeDuMi` - Free
- `LMI Lab` - in Matlab Robust Control Toolbox (in Computer lab)
- `YALMIP` - A nice front end for several solvers (Free)

The Schur complement

Before we get to the main result, recall the Schur complement.

Theorem 17 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n\times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} > 0$
2. $Q > 0$ *and* $M - RQ^{-1}R^T > 0$

A commonly used property of positive matrices.
Also Recall: If $X > 0$,

- then $X - \epsilon I > 0$ for ϵ sufficiently small.

The KYP Lemma (AKA: The Bounded Real Lemma)

The most important theorem in this class.

Lemma 18.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Can be used to calculate the H_∞-norm of a system

- Originally used to solve LMI's using graphs. (Before Computers)
- Now used directly instead of graphical methods like Bode.

The feasibility constraints are linear

- Can be combined with other methods.

The KYP Lemma

Proof.

We will only show that ii) implies i). The other direction requires the Hamiltonian, which we have not discussed.

- We will show that if $y = Gu$, then $\|y\|_{L_2} \leq \gamma \|u\|_{L_2}$.
- From the 1 x 1 block of the LMI, we know that $A^TX + XA < 0$, which means A is Hurwitz.
- Because the inequality is strict, there exists some $\epsilon > 0$ such that

$$\begin{aligned}
&\begin{bmatrix} A^TX + XA & XB \\ B^TX & -(\gamma - \epsilon)I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \\
&= \begin{bmatrix} A^TX + XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} < 0
\end{aligned}$$

- Let $y = Gu$. Then the state-space representation is

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
\dot{x}(t) &= Ax(t) + Bu(t) \qquad\qquad x(0) = 0
\end{aligned}$$

The KYP Lemma

Proof.

- Let $V(x) = x^T X x$. Then the LMI implies

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left[\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} y^T y$$

$$= x^T (A^T X + XA) x + x^T X B u + u^T B^T X x - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= (Ax + Bu)^T X x + x^T X (Ax + Bu) - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2$$

$$= \dot{V}(x(t)) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2 < 0$$

The KYP Lemma

Proof.

- Now we have $\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2 < 0$
- Integrating in time, we get

$$\int_0^T \left(\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2\right)dt$$
$$= V(x(T)) - V(x(0)) - (\gamma - \epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2\Big) dt < 0$$

- Because A is Hurwitz, $\lim_{t\to\infty} x(t) = 0$.
- Hence $\lim_{t\to\infty} V(x(t)) = 0$.
- Likewise, because $x(0) = 0$, we have $V(x(0)) = 0$.

□

The KYP Lemma

Proof.

- Since $V(x(0)) = V(x(\infty)) = 0$,

$$\lim_{T\to\infty}\left[\dot V(x(T)) - \dot V(x(0)) - (\gamma-\epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2\Big) dt\right]$$
$$= 0 - 0 - (\gamma-\epsilon)\int_0^\infty \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^\infty \|y(t)\|^2 dt$$
$$= -(\gamma-\epsilon)\|u\|_{L_2}^2 + \frac{1}{\gamma}\|y\|_{L_2}^2 dt < 0$$

- Thus

$$\|y\|_{L_2}^2 dt < (\gamma^2 - \epsilon\gamma)\|u\|_{L_2}^2$$

- By definition, this means $\|G\|_{H_\infty}^2 \le (\gamma^2 - \epsilon\gamma) < \gamma^2$ or

$$\|G\|_{H_\infty} < \gamma$$

□

A Variation on the KYP lemma is the positive-real lemma

Lemma 19.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- G *is passive. i.e.* $(\langle u, Gu\rangle_{L_2} \geq 0)$.
- *There exists a* $P > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0$$