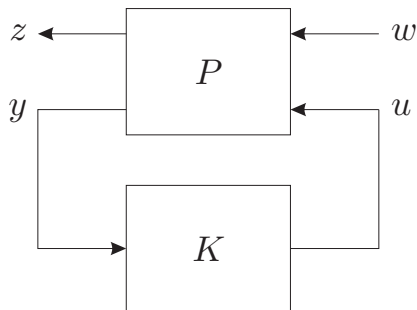


Modern Optimal Control

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Lecture 20: Optimal Full-State Feedback Control

Recall: Linear Fractional Transformation



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where} \quad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

Controller:

$$u = Ky \quad \text{where} \quad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$$

Optimal Control

Choose K to minimize

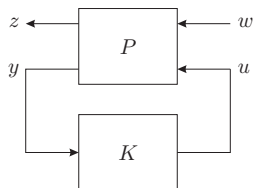
$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \left[\begin{array}{cc} A & 0 \\ 0 & A_K \end{array} \right] + \left[\begin{array}{cc} B_2 & 0 \\ 0 & B_K \end{array} \right] \left[\begin{array}{cc} I & -D_K \\ -D_{22} & I \end{array} \right]^{-1} \left[\begin{array}{cc} 0 & C_K \\ C_2 & 0 \end{array} \right] & \begin{array}{c} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{array} \\ \hline \left[\begin{array}{cc} C_1 & 0 \end{array} \right] + \left[\begin{array}{cc} D_{12} & 0 \end{array} \right] \left[\begin{array}{cc} I & -D_K \\ -D_{22} & I \end{array} \right]^{-1} \left[\begin{array}{cc} 0 & C_K \\ C_2 & 0 \end{array} \right] & D_{11} + D_{12} D_K Q D_{21} \end{array} \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

Optimal Full-State Feedback Control



For the full-state feedback case, we consider a controller of the form

$$u(t) = Fx(t)$$

Controller:

$$u = Ky \quad \text{where} \quad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right]$$

Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where} \quad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right]$$

Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P}, \hat{K}) = \left[\begin{array}{c|c} A + B_2F & B_1 \\ \hline C_1 + D_{12}F & D_{11} \end{array} \right]$$

By the KYP lemma, $\|\underline{S}(\hat{P}, \hat{K})\|_{H_\infty} < \gamma$ if and only if there exists some $X > 0$ such that

$$\begin{aligned} & \left[\begin{array}{cc} (A + B_2F)^T X + X(A + B_2F) & XB_1 \\ B_1^T X & -\gamma I \end{array} \right] \\ & + \frac{1}{\gamma} \left[\begin{array}{c} (C_1 + D_{12}F)^T \\ D_{11}^T \end{array} \right] \left[\begin{array}{cc} (C_1 + D_{12}F) & D_{11} \end{array} \right] < 0 \end{aligned}$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Schur Complement

The KYP condition is

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Recall the Schur Complement

Theorem 1 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n \times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} < 0$
2. $Q < 0$ and $M - RQ^{-1}R^T < 0$

In this case, let $Q = -\frac{1}{\gamma}I < 0$,

$$M = \begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} \quad R = \begin{bmatrix} C & D \end{bmatrix}^T$$

Note we are making the LMI **Larger**.

Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the

Full-State Feedback Condition

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 & (C_1 + D_{12} F)^T \\ B_1^T X & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F .

Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

Lemma 2 (KYP Dual).

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \leq \gamma$.
- There exists a $Y > 0$ such that

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

Proof.

Let $X = Y^{-1}$. Then

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0 \quad \text{and} \quad Y > 0$$

if and only if $X > 0$ and

$$\begin{aligned} & \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

By the Schur complement this is equivalent to

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

By the KYP lemma, this equivalent to $\|G\|_{H_\infty} \leq \gamma$. □

Full-State Feedback Optimal Control

We can now apply this result to the state-feedback problem.

Theorem 3.

The following are equivalent:

- There exists an F such that $\|\underline{S}(P, K(0, 0, 0, F))\|_{H_\infty} \leq \gamma$.
- There exist $Y > 0$ and Z such that

$$\begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ & B_1^T & D_{11}^T \\ & C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

Then $F = ZY^{-1}$.

Full-State Feedback Optimal Control

Proof.

Suppose there exists an F such that $\|\underline{S}(P, K(0,0,0, F))\|_{H_\infty} \leq \gamma$. By the Dual KYP lemma, this implies there exists a $Y > 0$ such that

$$\begin{bmatrix} Y(A + B_2F)^T + (A + B_2F)Y & B_1 & Y(C_1 + D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0$$

Let $Z = FY$. Then

$$\begin{aligned} & \begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix} \\ = & \begin{bmatrix} YA^T + YF^T B_2^T + AY + B_2FY & B_1 & YC_1^T + YF^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix} \\ = & \begin{bmatrix} Y(A + B_2F)^T + (A + B_2F)Y & B_1 & Y(C_1 + D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

Full-State Feedback Optimal Control

Proof.

Now suppose there exists a $Y > 0$ and Z such that

$$\begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ & B_1^T & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

Let $F = ZY^{-1}$. Then

$$\begin{aligned} & \begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)Y & B_1 & Y(C_1 + D_{12} F)^T \\ & B_1^T & D_{11}^T \\ (C_1 + D_{12} F)Y & D_{11} & -\gamma I \end{bmatrix} \\ = & \begin{bmatrix} YA^T + YF^T B_2^T + AY + B_2 FY & B_1 & YC_1^T + YF^T D_{12}^T \\ & B_1^T & D_{11}^T \\ C_1 Y + D_{12} FY & D_{11} & -\gamma I \end{bmatrix} \\ = & \begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ & B_1^T & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0 \end{aligned}$$

Full-State Feedback Optimal Control

Therefore the following optimization problems are equivalent

Form A

$$\min_F \|\underline{S}(P, K(0, 0, 0, F))\|_{H_\infty}$$

Form B

$\min_{\gamma, Y, Z} \gamma :$

$$\begin{bmatrix} -Y & 0 & 0 & 0 \\ 0 & YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ 0 & B_1^T & -\gamma I & D_{11}^T \\ 0 & C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The optimal controller is given by $F = ZY^{-1}$.

Next Lecture: Optimal Output Feedback