Modern Optimal Control

Matthew M. Peet
Arizona State University

Lecture 22: H_2, LQG and LGR

Conclusion

To solve the H_∞-optimal state-feedback problem, we solve

$$\min_{\gamma, X_1, Y_1, A_n, B_n, C_n, D_n} \gamma \qquad \text{such that}$$

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X B_1 + B_n D_{21}]^T & -\gamma I & \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

Conclusion

Then, we construct our controller using

$$
\begin{aligned}
D_K &= (I + D_{K2}D_{22})^{-1}D_{K2}\\
B_K &= B_{K2}(I - D_{22}D_K)\\
C_K &= (I - D_K D_{22})C_{K2}\\
A_K &= A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\end{aligned}
$$

where

$$
\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1AY_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2Y_1 & I \end{bmatrix}^{-1}.
$$

and where X_2 and Y_2 are any matrices which satisfy $X_2Y_2 = I - X_1Y_1$.

- e.g. Let $Y_2 = I$ and $X_2 = I - X_1Y_1$.
- The optimal controller is NOT uniquely defined.
- Don't forget to check invertibility of $I - D_{22}D_K$

Conclusion

The H_∞-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

Minimizes the effect of external input (w) on external output (z).

$$\|z\|_{L_2} \le \|\underline{\mathsf{S}}(P,K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain

H_2-optimal control
Motivation

H_2-optimal control minimizes the H_2-norm of the transfer function.

- The H_2-norm has no direct interpretation.

$$\|G\|_{H_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{G}^*(\imath\omega)\hat{G}(\imath\omega))d\omega$$

Motivation: Assume external input is Gaussian noise with spectral density S_w

$$E[w(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{S}_w(\imath\omega))d\omega$$

Theorem 1.

For an LTI system P, if w is noise with spectral density $\hat{S}_w(\imath\omega)$ and $z = Pw$, then z is noise with density

$$\hat{S}_z(\imath\omega) = \hat{P}(\imath\omega)\hat{S}(\imath\omega)\hat{P}(\imath\omega)^*$$

Then the output $z = Pw$ has signal variance (Power)

$$E[z(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{G}^*(\imath\omega)S(\imath\omega)\hat{G}(\imath\omega))d\omega$$
$$\leq \|S\|_{H_\infty}\|G\|_{H_2}^2$$

If the input signal is white noise, then $\hat{S}(\imath\omega) = I$ and

$$E[z(t)^2] = \|G\|_{H_2}^2$$

If the noise is colored, then we can still use the same approach to H_2 optimal control. Let $\hat{H}(\imath\omega)\hat{H}(\imath\omega)^* = \hat{S}_w(\imath\omega)$. Then design an H_2-optimal controller for the plant

$$\hat{P}_s(s) = \begin{bmatrix} \hat{P}_{11}(s)\hat{H}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s)\hat{H}(s) & \hat{P}_{22}(s) \end{bmatrix}$$

Now, using the controller and filtered plant,

$$\underline{\mathsf{S}}(P_s, K)(s) = P_{11}H + P_{12}(I - KP_{22})^{-1}KP_{21}H = \underline{\mathsf{S}}(P, K)H$$

For noise with spectral density S_w, let $z = \underline{\mathsf{S}}(P, K)w$. Then if S_z is the spectral density of z, we have

$$\begin{aligned} S_z(s) &= \underline{\mathsf{S}}(P, K)(s)\hat{S}_w(s)\underline{\mathsf{S}}(P, K)(s)^* \\ &= \underline{\mathsf{S}}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\underline{\mathsf{S}}(P, K)(s)^* = \underline{\mathsf{S}}(P_s, K)(s)\hat{S}(P_s, K)(s)^* \end{aligned}$$

and hence $E[z(t)^2] = \|\hat{S}(P_s, K)\|_{H_2}^2 = \gamma$. Thus minimization of $\|\hat{S}(P_s, K)\|_{H_2}^2$ achieves the optimal system response to noise with density $\hat{S}_w$.

H_2-optimal control

Theorem 2.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. A *is Hurwitz and* $\|\hat{P}\|_{H_2} < \gamma$.
2. *There exists some* $X > 0$ *such that*

$$\operatorname{trace} CXC^T < \gamma$$
$$AX + XA^T + BB^T < 0$$

H_2-optimal control

Proof.

Suppose A is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Controllability Grammian is defined as

$$X_c = \int_0^\infty e^{At} B B^T e^{A^T} dt$$

Now recall the Laplace transform

$$\begin{aligned}
\left(\Lambda e^{At}\right)(s) &= \int_0^\infty e^{At} e^{-ts} dt \\
&= \int_0^\infty e^{-(sI-A)t} dt \\
&= -(sI-A)^{-1} e^{-(sI-A)t} dt \Big|_{t=0}^{t=-\infty} \\
&= (sI-A)^{-1}
\end{aligned}$$

Hence $\left(\Lambda C e^{At} B\right)(s) = C(sI-A)^{-1}B$. □

H_2-optimal control

Proof.

$\left(\Lambda C e^{At} B\right)(s) = C(sI - A)^{-1}B$ implies

$$\begin{aligned}
\|\hat{P}\|_{H_2}^2 &= \|C(sI - A)^{-1}B\|_{H_2}^2 \\
&= \frac{1}{2\pi}\int_0^\infty \text{Trace}((C(\imath\omega I - A)^{-1}B)^*(C(\imath\omega I - A)^{-1}B))d\omega \\
&= \frac{1}{2\pi}\int_0^\infty \text{Trace}((C(\imath\omega I - A)^{-1}B)(C(\imath\omega I - A)^{-1}B)^*)d\omega \\
&= \text{Trace}\int_{-\infty}^\infty C e^{At} B B^* e^{A^* t} C^* dt \\
&= \text{Trace} C X_c C^T
\end{aligned}$$

Thus $X_c \geq 0$ and Trace$C X_c C^T = \|\hat{P}\|_{H_2}^2 < \gamma$. □

Proof.

Likewise Trace$B^T X_o B = \|\hat{P}\|_{H_2}^2$. To show that we can take strict the inequality $X > 0$, we simply let

$$X = \int_0^\infty e^{At} \left(BB^T + \epsilon I\right) e^{A^T} dt$$

for sufficiently small $\epsilon > 0$. Furthermore, we already know the controllability grammian X_c and thus X_ϵ satisfies the Lyapunov inequality.

$$A^T X_\epsilon + X_\epsilon A + BB^T < 0$$

These steps can be reversed to obtain necessity. □

H_2-optimal control

Full-State Feedback

Lets consider the full-state feedback problem

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

- D_{12} is the weight on control effort.
- $D_{11} = 0$ is neglected as the feed-through term.
- $C_2 = I$ as this is state-feedback.

$$\hat{K}(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right]$$

Theorem 3.

The following are equivalent.

1. $\|S(K,P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ *for some* Z *and* $X > 0$ *where*

$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0$$

$$\textit{Trace} \left[C_1 X + D_{12} Z \right] X^{-1} \left[C_1 X + D_{12} Z \right] < \gamma$$

However, this is nonlinear, so we need to reformulate using the Schur Complement.

H_2-optimal control

Applying the Schur Complement gives the alternative formulation convenient for control.

Theorem 4.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$*. Then the following are equivalent.*

1. A *is Hurwitz and* $\|\hat{P}\|_{H_2} < \gamma$*.*
2. *There exists some* $X, Z > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} X & C^T \\ C & Z \end{bmatrix} > 0, \qquad \textit{Trace} Z < \gamma$$

Theorem 5.

The following are equivalent.

1. $\|S(K,P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ *for some* Z *and* $X > 0$ *where*

$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0$$

$$\begin{bmatrix} X & (C_1 X + D_{12} Z)^T \\ C_1 X + D_{12} Z & W \end{bmatrix} > 0$$

$$\text{Trace} W < \gamma$$

Thus we can solve the H_2-optimal static full-state feedback problem.

H_2-optimal control

Relationship to LQR

Thus minimizing the H_2-norm minimizes the effect of white noise on the power of the output noise.

- This is why H_2 control is often called Least-Quadratic-Gaussian (LQG).

LQR:

- Full-State Feedback
- Choose K to minimize the cost function

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

subject to dynamic constraints

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

To solve the LQR problem, let

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$
- $B_2 = B$ and $B_1 = I$.

So that

$$\underline{\mathsf{S}}(\hat{P}, \hat{K}) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C_1 + D_{12} K & D_{11} \end{array}\right] = \left[\begin{array}{c|c} A + BK & I \\ \hline \begin{matrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K \end{matrix} & 0 \end{array}\right]$$

And solve the H_2 full-state feedback problem. Then if

$$\dot{x}(t) = A_{CL} x(t) = (A + BK) x(t) = A x(t) + B u(t)$$
$$u(t) = K x(t), \qquad x(0) = x_0$$

Then $x(t) = e^{A_{CL} t} x_0$

If

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

then $x(t) = e^{A_{CL}t}x_0$ and

$$\int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)dt = \int_0^\infty x_0^T e^{A_{CL}^T t}(Q + K^T RK)e^{A_{CL}t}x_0 dt$$
$$= \text{Trace} \int_0^\infty x_0^T e^{A_{CL}^T t} \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix}^T \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix} e^{A_{CL}t}x_0 dt$$
$$= \|x_0\|^2 \text{Trace} \int_0^\infty B_1 e^{A_{CL}^T t}(C_1 + D_{12}K)^T (C_1 + D_{12}K)e^{A_{CL}t}B_1^T dt$$
$$= \|x_0\|^2 \|S(K,P)\|_{H_2}^2$$

Thus LQR reduces to a special case of H_2 static state-feedback.

H_2-optimal output feedback control

Theorem 6 (Lall).

The following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma$.
- *There exist* $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ *such that* $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X B_1 + B_n D_{21}]^T & -\gamma I \end{bmatrix} < 0,$$

$$\begin{bmatrix} X_1 & I & *^T \\ I & Y_1 & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z \end{bmatrix} > 0,$$

$$D_{11} + D_{12} D_n D_{21} = 0, \qquad \mathrm{trace}(Z) < \gamma$$

H_2-optimal output feedback control

As before, the controller can be recovered as

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}$$

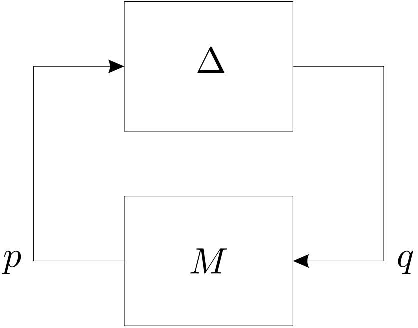
for any full-rank X_2 and Y_2 such that

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

To find the actual controller, we use the identities:

$$\begin{aligned} D_K &= (I + D_{K2} D_{22})^{-1} D_{K2} \\ B_K &= B_{K2}(I - D_{22} D_K) \\ C_K &= (I - D_K D_{22}) C_{K2} \\ A_K &= A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K \end{aligned}$$

Before we finish, let us briefly touch on the use of LMIs in Robust Control.



Questions:

- Is $\underline{\mathsf{S}}(\Delta, M)$ stable for all $\Delta \in \boldsymbol{\Delta}$?
- Determine

$$\sup_{\Delta \in \boldsymbol{\Delta}} \|\underline{\mathsf{S}}(\Delta, M)\|_{H_\infty}.$$