Modern Optimal Control

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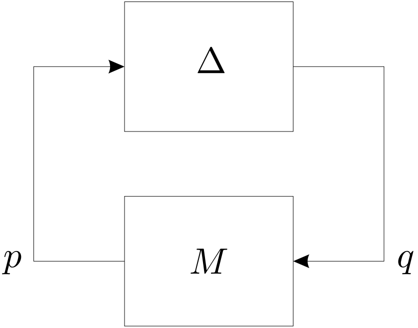
Lecture 23: Robust Control

Robust Control

Before we finish, let us briefly touch on the use of LMIs in Robust Control.



Questions:

- Is $\underline{\mathsf{S}}(\Delta, M)$ stable for all $\Delta \in \boldsymbol{\Delta}$?
- Determine

$$\sup_{\Delta \in \boldsymbol{\Delta}} \|\underline{\mathsf{S}}(\Delta, M)\|_{H_\infty}.$$

Suppose we have the system M

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Definition 1.

We say the pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if $(I - M_{22}\Delta)$ is invertible for all $\Delta \in \mathbf{\Delta}$.

$$S_l(M, \Delta) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}$$

The structure of $\mathbf{\Delta}$ makes a lot of difference. e.g.

- **Unstructured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$

- **Structured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta_1, \Delta_2, \cdots \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$

- **Unstructured, Parametric, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

- **Unstructured, Parametric, polytopic:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\, \alpha_i \ge 0,\, \sum_i \alpha_i \le 1\}$$

Let's consider a simple question: *Additive Uncertainty*.

$$M_{22} = 0, \quad M_{12} = M_{21} = I$$

Question: Is $\dot{x} = A(t)x(t)$ stable if $A(t) \in \mathbf{\Delta}$ for all $t \geq 0$.

Definition 2 (Quadratic Stability).

$\dot{x} = A(t)x(t)$ is **Quadratically Stable** for $A(t) \in \mathbf{\Delta}$ if there exists some $P > 0$ such that

$$A^T P + PA < 0 \qquad \text{for all } A \in \mathbf{\Delta}$$

Theorem 3.

If $\dot{x} = A(t)x(t)$ is Quadratically Stable, then it is stable for $A \in \mathbf{\Delta}$.

We examine this problem for:

- **Parametric, Polytopic Uncertainty:**

$$\boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i A_i,\ \alpha_i \geq 0,\ \sum_i \alpha_i \leq 1\}$$

Parametric, Polytopic Uncertainty

For the polytopic case, we have the following result

Theorem 4 (Quadratic Stability).

Let

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\, \alpha_i \geq 0,\, \sum_i \alpha_i \leq 1\}$$

Then $\dot{x}(t) = A(t)x(t)$ *is quadratically stable for all* $A \in \mathbf{\Delta}$ *if and only if there exists some* $P > 0$ *such that*

$$A_i^T P + P A_i < 0 \qquad \textit{for } i = 1, \cdots ,$$

Thus quadratic stability of systems with polytopic uncertainty is equivalent to an LMI.

Parametric, Norm-Bounded Uncertainty

A more complex uncertainty set is:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + M p(t), \qquad & p(t) &= \Delta(t) q(t), \\ q(t) &= N x(t) + Q p(t), \qquad & \Delta &\in \mathbf{\Delta} \end{aligned}$$

- **Parametric, Norm-Bounded Uncertainty:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \,:\, \|\Delta\| \le 1\}$$

Parametric, Norm-Bounded Uncertainty

Quadratic Stability: There exists a $P > 0$ such that

$P(A_0x(t)+Mp)+(A_0x(t)+Mp)^TP < 0$ for all $p \in \{p : p = \Delta q, q = Nx+Qp\}$

Theorem 5.

The system

$$\dot{x}(t) = A_0x(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} : \|\Delta\| \leq 1\}$$

is quadratically stable if and only if there exists some $P > 0$ *such that*

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} A^TP + PA & PM \\ M^TP & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -N^TN & -N^TQ \\ -Q^TN & I - Q^TQ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 0 \right\}$$

Parametric, Norm-Bounded Uncertainty

If.

If

$$\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ M^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 0 \right\}$$

then

$$x^T P(Ax + My) + (Ax + My)^T Px < 0$$

for all x, y such that

$$\|Nx + Qy\|^2 \leq \|y\|^2$$

Therefore, since $p = \Delta q$ implies $\|p\| \leq \|q\|$, we have quadratic stability.
The *only if* direction is similar.

□

S-procedure to the rescue!
The S-procedure asks the question:

- Is $z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$?

Corollary 6 (S-Procedure).

$z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$ if there exists a $\tau \geq 0$ such that $F - \tau G \succeq 0$.

The S-procedure is **Necessary** if $\{x : x^T G x > 0\} \neq \emptyset$.

Parametric, Norm-Bounded Uncertainty

Theorem 7.

The system

$$\dot{x}(t) = A_0 x(t) + Mp(t), \qquad p(t) = \Delta(t) q(t),$$
$$q(t) = Nx(t) + Qp(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

is quadratically stable if and only if there exists some $\mu \ge 0$ *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}$$

These approaches can be readily extended to controller synthesis.

Quadratic Stability

Consider Quadratic Stability in Discrete-Time: $x_{k+1} = S_l(M, \Delta)x_k$.

Definition 8.

$(S_l, \mathbf{\Delta})$ is QS if

$$S_l(M,\Delta)^T P S_l(M,\Delta) - P < 0 \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

Theorem 9 (Packard and Doyle).

Let $M \in \mathbb{R}^{(n+m)\times(n+m)}$ be given with $\rho(M_{11}) \le 1$ and $\sigma(M_{22}) < 1$. Then the following are equivalent.

1. *The pair $(M, \Delta = \mathbb{R}^{m\times m})$ is quadratically stable.*
2. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is quadratically stable.*
3. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is robustly stable.*

The Structured Singular Value

For the case of structured parametric uncertainty, we define the structured singular value.

$$\mathbf{\Delta} = \{\Delta = \text{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) : \delta_i \in \mathbb{F}, \Delta \in \mathbb{F}^{n_k \times n_k}\}$$

- δ and Δ represent unknown parameters.
- s is the number of scalar parameters.
- f is the number of matrix parameters.

Definition 10.

Given system $M \in \mathcal{L}(L_2)$ and set $\mathbf{\Delta}$ as above, we define the **Structured Singular Value** of $(M, \mathbf{\Delta})$ as

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf_{\substack{\Delta \in \mathbf{\Delta} \\ I - M_{22}\Delta \text{ is singular}}} \|\Delta\|}$$

The Structured Singular Value

Theorem 11.

Let

$$\mathbf{\Delta}_n = \{\Delta \in \mathbf{\Delta},\ \|\Delta\| \leq \mu(M, \mathbf{\Delta})\}.$$

Then the pair $(M, \mathbf{\Delta}_n)$ *is robustly stable.*

Conclusion

THE END