LMI Methods in Optimal and Robust Control

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Lecture 03: Relaxations, Duality, Cones, Positive Matrices, LMIs

Linear Algebra Review: Matrix Decompositions

Definition 1 (Eigenvalue/Eigenvector Decomposition (EED)).

$P \in \mathbb{R}^{n\times n}$, is real non-defective (diagonalizable) if there exists invertible $T \in \mathbb{R}^{n\times n}$ and a diagonal $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$P = T\Lambda T^{-1}$$

Not all Matrices have an EED: e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **Matlab command:** `eig`

Theorem 1 (Jordan Normal Form).

For any $P \in \mathbb{R}^{n\times n}$, there exist Jordan blocks, J_i (each with associated eigenvalue λ_i), and an invertible $T \in \mathbb{R}^{n\times n}$ such that

$$P = TJT^{-1}, J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix} \quad J_i = \lambda_i I + N, \quad N = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

Matlab command: `jordan` $N \in \mathbb{R}^{n\times n}$ is *Nilpotent*, so $N^n = 0$.

Lecture 03

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Theorem 1 (Jordan Normal Form).

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`jordan` requires symbolic math toolbox and is not reliable.

Linear Algebra Review: Symmetric and Unitary Matrices

Definition 2.

A square matrix $P \in \mathbb{R}^{n \times n}$ is **Symmetric**, denoted $P \in \mathbb{S}^n$ if $P = P^T$.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & 3 \\ *^T & 4 & 5 \\ *^T & *^T & 6 \end{bmatrix}$$

- Symmetric Matrices have **Real** Eigenvalues: $\{\lambda \,:\, Px = \lambda x,\ x \in \mathbb{R}^n\} \subset \mathbb{R}$

Definition 3.

A matrix U is **Unitary** (orthogonal) if $U^{-1} = U^T$.

Unitary Matrices have the pleasant property that $\|Ux\| = \|x\|$ for any $x \in \mathbb{R}^n$.

- e.g. Rotation Matrices $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Symmetric Matrices can be diagonalized by a Unitary matrix.

$$P = U \Lambda U^T \quad \text{or} \quad U^T P U = \Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \quad (\lambda_i \text{ are eigenvalues})$$

Linear Algebra Review: Singular Value Decomposition

For ANY non-symmetric matrices, $P = U\Sigma V^T$, with U, V unitary and $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$ diagonal, positive.

- This is called the **Singular Value Decomposition (SVD)**.
- $\sigma_i \geq 0$ (**Singular Values**) are the square roots of the eigenvalues of $P^T P$.
- The maximum σ_i is denoted $\bar{\sigma}(P)$. Note that $\bar{\sigma}(P) = \max_x \frac{\|Px\|}{\|x\|}$.

Matlab Code: `> [U,S,V]=svd([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}11.34 & 0 & 0\\ 0 & .52 & 0\\ 0 & 0 & .17\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}-.33 & .74 & -.59\\ -.59 & .33 & .74\\ -.74 & -.59 & -.33\end{bmatrix}^T}_{V^T}$$

NOTE: This is not quite the same as Diagonalization! Unless....

Matlab Code: `> [U,S]=schur([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}11.34 & 0 & 0\\ 0 & -.52 & 0\\ 0 & 0 & .17\end{bmatrix}}_{\Lambda}\underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}^T}_{U^T}$$

Linear Algebra Review: More Matrix Decompositions

Lemma 4 (QR Decomposition).

For any matrix $P \in \mathbb{R}^{n \times m}$, there exists a matrix Q where $Q^T Q = I$ and upper triangular matrix, R such that $P = QR$.

Lemma 5 (Schur Decomposition).

For any square matrix $P \in \mathbb{R}^{n \times n}$, there exists a unitary matrix U and upper triangular matrix, U such that $P = VUV^T$.

Lemma 6 (LU Decomposition).

For any nonsingular square matrix $P \in \mathbb{R}^{n \times n}$, there exist a permutation matrix, T; lower triangular matrix, L; and upper triangular matrix, U such that

$$TP = LU$$

Lemma 7 (Cholesky Decomposition).

Suppose $P \in \mathbb{S}^{n \times n}$ has all strictly positive eigenvalues. Then there exist a lower triangular matrix, L such that

$$P = LL^T$$

Matrix Positivity - Definition

Try not to define positivity using eigenvalues. (Eigenvalues don't add)

Definition 8.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Definition 9.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \qquad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

How to Tell if a Matrix is Positive?

Lemma 10.

$P \in \mathbb{S}^n$ is positive definite if and only if all its eigenvalues are positive.

In this case, the SVD and Unitary (Schur) Diagonalization are the same.

$$\begin{bmatrix}4 & 1 & 2\\ 1 & 5 & 3\\ 2 & 3 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.37 & .82 & -.44\\ -.58 & -.58 & -.58\\ -.73 & .04 & .69\end{bmatrix}}_{U} \underbrace{\begin{bmatrix}9.4 & 0 & 0\\ 0 & 3.4 & 0\\ 0 & 0 & 2.2\end{bmatrix}}_{\Lambda=\Sigma} \underbrace{\begin{bmatrix}-.37 & .82 & -.44\\ -.58 & -.58 & -.58\\ -.73 & .04 & .69\end{bmatrix}^T}_{U^T=V^T}$$

Lemma 11 (Sylvester's Criterion).

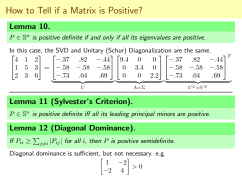
$P \in \mathbb{S}^n$ is positive definite iff all its leading principal minors are positive.

Lemma 12 (Diagonal Dominance).

If $P_{ii} \ge \sum_{j\neq i} |P_{ij}|$ for all i, then P is positive semidefinite.

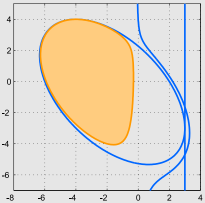
Diagonal dominance is sufficient, but not necessary. e.g.

$$\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix} > 0$$



Use Sylvester's Criterion to determine for which x, y

$$\begin{bmatrix} 3-x & -(x+y) & 1 \\ -(x+y) & 4-y & 0 \\ 1 & 0 & -x \end{bmatrix} \ge 0.$$



The leading principal minors are

$$\det(3-x) \ge 0 \quad \det\begin{bmatrix} 3-x & -(x+y) \\ -(x+y) & 4-y \end{bmatrix} \ge 0 \quad \det\begin{bmatrix} 3-x & -(x+y) & 1 \\ -(x+y) & 4-y & 0 \\ 1 & 0 & -x \end{bmatrix} \ge 0$$

Which gives the following inequalities

$$f_1(x,y) = 3 - x \ge 0 \qquad f_2(x,y) = (3-x)(4-y) - (x+y)^2 \ge 0$$
$$f_3(x,y) = -x((3-x)(4-y) - (x+y)^2) - (4-y) \ge 0$$

Although f_3 is not convex, the feasible set *is* convex, as expected.

Facts about Positive Matrices

Fact: If T is invertible, then $P > 0$ is equivalent to $T^T PT > 0$.

- $P > 0 \rightarrow (Tx)^T P(Tx) = x^T T^T PTx > 0$
- $T^T PT > 0 \rightarrow (T^{-1}x)^T T^T PT(T^{-1}x) = x^T Px > 0$

Fact: A Positive Definite matrix is invertible: $P^{-1} = U\Sigma^{-1}U^T$.
Fact: The inverse of a positive definite matrix is positive definite: $\Sigma^{-1} > 0$
Fact: For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ where $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

$$P^{\frac{1}{2}} = U\Sigma^{\frac{1}{2}}U^T > 0 \qquad P^{\frac{1}{2}}P^{\frac{1}{2}} = U\Sigma^{\frac{1}{2}}U^T U\Sigma^{\frac{1}{2}}U^T = U\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}U^T = U\Sigma U^T = P$$

Building Linear Matrix Inequalities

Fact: $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, implies both $X > 0$ and $Z > 0$.

Proof: True since $\begin{bmatrix} 0 \\ z \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} > 0$ and $\begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$

Fact: $X > 0$ and $Z > 0$ is equivalent to $\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} > 0$.

Proof: True since $x^T X x > 0$ and $z^T Z z > 0$ implies
$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = x^T X x + z^T Z z > 0$.

Theorem 13 (Schur Complement).

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X - Y Z^{-1} Y^T & 0 \\ 0 & Z \end{bmatrix} > 0$$

Diagonal Dominance: If X and Z are big enough, Y doesn't matter.

Leftover Factoids on Positive Matrices

Things which are true:

- $P > 0$ and $Q > 0$ implies $P + Q > 0$.
- $P > 0$ implies $\mu P > 0$ for any positive scalar $\mu > 0$.
- $M^T M \geq 0$ for any matrix, M.
- $P > 0$ implies $M^T P M > 0$ if nullspace of M is empty.

Things which are NOT TRUE (Fallacies):

1. A has positive eigenvalues implies $A + A^T > 0$.
2. A has positive eigenvalues implies $x^T A x > 0$ for all $x \neq 0$.
3. $P > 0$ implies $T P T^{-1} > 0$.
4. $P > 0$ and $Q > 0$ implies $PQ > 0$.
5. If $P > 0$ and T has positive eigenvalues, then $T^T P + P T > 0$
6. If $P > 0$, $Q > 0$, and $R > 0$, then $\begin{bmatrix} P & Q \\ Q & R \end{bmatrix} > 0$.

Semidefinite Programming - Dual Form

$$\begin{aligned} \text{minimize} \quad & \operatorname{trace} CX \\ \text{subject to} \quad & \operatorname{trace} F_i X = -c_i \qquad \text{for all } i \\ & X \succeq 0 \end{aligned}$$

- The variable X is a symmetric matrix
- $X \succeq 0$ is another way to say X is positive semidefinite
- The feasible set is the intersection of an affine set with the *positive semidefinite cone*

$$\left\{ X \in \mathbb{S}^n \mid X \succeq 0 \right\}$$

Recall $\operatorname{trace} CX = \sum_{i,j} C_{i,j} X_{j,i}$.

SDPs with Explicit Variables - Primal Form

We can also explicitly parametrize the affine set to give

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & C + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *Linear Matrix Inequality (LMI)*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Linear Matrix Inequalities

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.
Common Form:

$$\text{Find } X :$$
$$\sum_i A_i X B_i + Q > 0$$

There are several very efficient **LMI/SDP Solvers** which interface with YALMIP:

- **SeDuMi**
  - Fast, but somewhat unreliable.
  - Link: http://sedumi.ie.lehigh.edu/
- **LMI Lab** (Part of Matlab's Robust Control Toolbox)
  - Universally disliked, but you already have it.
  - Link: http://www.mathworks.com/help/robust/lmis.html
- **MOSEK** (commercial, but free academic licenses available)
  - Probably the most reliable
  - Link: https://www.mosek.com/resources/academic-license