LMI Methods in Optimal and Robust Control

Matthew M. Peet
Arizona State University

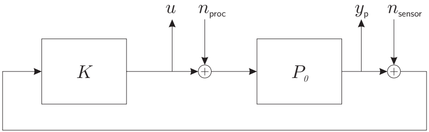
Lecture 12: Modeling Uncertainty and Robustness

Robust Control: Dealing with Uncertainty

The Known Unknowns

CASE 1: External Disturbances

- The most benign source of uncertainty.
- Finite Energy (L_2-norm bounded).
- H_∞ optimal control minimizes the effect of these uncertainties.



Benign Sources:

- Vibrations, Wind, 60 Hz noise
- Initial Conditions
- Sensor Noise
- Changes in Reference Signal

Not-So-Benign Sources:

- Higher-Order Dynamics
- Nonlinearity (Saturation)
- Delay
- Modeling Errors (Parametric vs. Structural)
- Model Reduction
- Logical Switching

The **Not-So-Benign Sources** describe uncertainty in the *System* (P).

- These can NOT be bounded apriori

The first step is to **Quantify** our uncertainty.

- How bad can it get?

We need to define the **Set** of possible Plants.

- $P \in \mathbf{P}$ where $\mathbf{P}$ is a set of possible plants.
- $\mathbf{P}$ can describe either finite or infinite possible systems.
- How do we parameterize $\mathbf{P}$

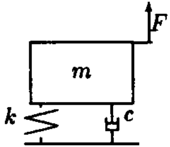
Original Problem:

$$\min_{K \in H_\infty} \|\underline{\mathsf{S}}(P, K)\|_{H_\infty}$$

Now we have to add a modifier:

$$\min_{K \in H_\infty} \gamma \,:\, \|\underline{\mathsf{S}}(P, K)\|_{H_\infty} \le \gamma \qquad \textbf{For All } P \in \mathbf{P}.$$

There are **Three Main Types of Parametric Uncertainty**



$$\ddot{y}(t) = \frac{c}{m}\dot{y}(t) + \frac{k}{m}y(t) = \frac{F(t)}{m}$$

- **Uncertainty in Parameters** c, k, m

Multiplicative Uncertainty

- $m = m_0(1 + \eta_m \delta_m)$
- $c = c_0(1 + \eta_c \delta_c)$
- $k = k_0(1 + \eta_k \delta_k)$

Where $\delta_m, \delta_c, \delta_k$ are bounded.

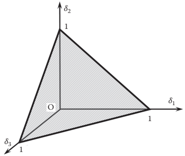
Additive Uncertainty

- $m = m_0 + \eta_m \delta_m$
- $c = c_0 + \eta_c \delta_c$
- $k = k_0 + \eta_k \delta_k$

Where $\delta_m, \delta_c, \delta_k$ are bounded.

Polytopic Uncertainty

$$\begin{bmatrix} m \\ c \\ k \end{bmatrix} \in \left\{ \begin{bmatrix} m \\ c \\ k \end{bmatrix} \; : \; \begin{bmatrix} m \\ c \\ k \end{bmatrix} = \sum_i \delta_i \begin{bmatrix} m_i \\ c_i \\ k_i \end{bmatrix}, \; \begin{matrix} \sum_i \delta_i = 1, \\ \delta_i \geq 0. \end{matrix} \right\}$$



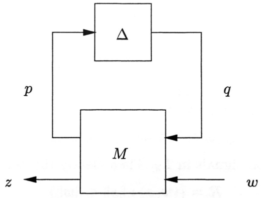
where $\begin{bmatrix} m_i & c_i & k_i \end{bmatrix}^T$ describe possible model parameters.

Linear-Fractional Representation

The first step is to isolate the unknowns from the knowns

The known part is the **Nominal System**, M:

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}$$



The unknown part is the **Uncertain System**, $q = \Delta p$

- For which we only know $\Delta \in \mathbf{\Delta}$.
- How to parameterize the *Set*: $\mathbf{\Delta}$?

As for the feedback interconnection, we have 3 equations:

$$p = M_{11}q + M_{12}w, \qquad z = M_{21}q + M_{22}w, \qquad q = \Delta p$$

Solving for q,

$$\begin{aligned} q &= \Delta p = \Delta M_{11} q + \Delta M_{12} w \\ &= (I - \Delta M_{11})^{-1} \Delta M_{12} w \end{aligned}$$

Then

$$z = M_{21}q + M_{22}w = \overbrace{\left(M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12}\right)}^{\bar{S}(M,\Delta)} w$$

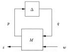
Recall that $\bar{S}(M, \Delta)$ is called the **Upper Star Product**.

Linear-Fractional Representation

The first step is to isolate the unknowns from the knowns

The known part is the **Nominal System**, M:

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}$$



The unknown part is the **Uncertain System**, $q = \Delta p$

- For which we only know $\Delta \in \boldsymbol{\Delta}$.
- How to parameterize the *Set*: $\boldsymbol{\Delta}$?

As for the feedback interconnection, we have 3 equations:

$$p = M_{11}q + M_{12}w, \qquad z = M_{21}q + M_{22}w, \qquad q = \Delta p$$

Solving for q,

$$\begin{aligned} q &= \Delta p = \Delta M_{11} q + \Delta M_{12} w \\ &= (I - \Delta M_{11})^{-1} \Delta M_{12} w \end{aligned}$$

Then

$$z = M_{21}q + M_{22}w = \overbrace{(M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12})}^{\bar{S}(M,\Delta)} w$$

Recall that $\bar{S}(M,\Delta)$ is called the **Upper Star Product**.

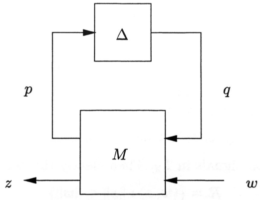
Note the algebraic use of systems.

- Δ and M_{ij} are subsystems, not matrices.
- This accounts for the lack of the time parameter, t, in the equations

Here we are using the 4-system representation of the nominal system. We can also do this using the 9-matrix representation, but recall the CL system is very complicated.

The **Nominal System**, M:

$$\begin{bmatrix} \dot{x}(t) \\ p(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_2 & B_1 \\ C_2 & D_{22} & D_{21} \\ C_1 & D_{12} & D_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \\ w(t) \end{bmatrix}$$



$\bar{S}(M, \Delta)$ is too complicated unless we
Assume *Static Uncertainty*: $q(t) = \Delta p(t)$:
Solving for q,

$$q(t) = \Delta(C_1 x(t) + D_{11} q(t) + D_{12} w(t))$$

$$\begin{aligned} q(t) &= (I - \Delta D_{11})^{-1} \Delta (C_1 x(t) + D_{12} w(t)) \\ &= (I - \Delta D_{11})^{-1} \Delta C_1 x(t) + (I - \Delta D_{11})^{-1} \Delta D_{12} w(t) \end{aligned}$$

Finally, we get

$$\dot{x}(t) = (A + B_1 (I - \Delta D_{11})^{-1} \Delta C_1) x(t) + (B_2 + B_1 (I - \Delta D_{11})^{-1} \Delta D_{12}) w(t)$$
$$z(t) = (C_2 + D_{21} (I - \Delta D_{11})^{-1} \Delta C_1) x(t) + (D_{22} + D_{21} (I - \Delta D_{11})^{-1} \Delta D_{12}) w(t)$$

Lecture 12

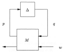
2023-10-23

Linear-Fractional Representation

Linear-Fractional Representation
State-Space Formulation

The **Nominal System**, M:

$$\begin{bmatrix}\dot{x}(t)\\ p(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A & B_2 & B_1\\ C_2 & D_{22} & D_{21}\\ C_1 & D_{12} & D_{11}\end{bmatrix}\begin{bmatrix}x(t)\\ q(t)\\ w(t)\end{bmatrix}$$



$\bar{S}(M,\Delta)$ is too complicated unless we
Assume *Static Uncertainty*: $q(t) = \Delta p(t)$:
Solving for q,

$$\begin{aligned} q(t) &= \Delta(C_1 x(t) + D_{11} q(t) + D_{12} w(t))\\ q(t) &= (I-\Delta D_{11})^{-1}\Delta(C_1 x(t) + D_{12} w(t))\\ &= (I-\Delta D_{11})^{-1}\Delta C_1 x(t) + (I-\Delta D_{11})^{-1}\Delta D_{12} w(t)\end{aligned}$$

Finally, we get

$$\dot{x}(t) = (A + B_1(I-\Delta D_{11})^{-1}\Delta C_1)x(t) + (B_2 + B_1(I-\Delta D_{11})^{-1}\Delta D_{12})w(t)$$
$$z(t) = (C_2 + D_{21}(I-\Delta D_{11})^{-1}\Delta C_1)x(t) + (D_{22} + D_{21}(I-\Delta D_{11})^{-1}\Delta D_{12})w(t)$$

- We are representing the LFT as a state-space equivalent representation, which may be easier to work with/understand - even though it involves more equations.
- Here we treat Δ as a matrix and not a system.

The CL system is

$$\bar{S}(M,\Delta) = \left[\begin{array}{c|c} A + B_1(I-\Delta D_{11})^{-1}\Delta C_1 & B_2 + B_1(I-\Delta D_{11})^{-1}\Delta D_{12}\\ \hline C_2 + D_{21}(I-\Delta D_{11})^{-1}\Delta C_1 & D_{22} + D_{21}(I-\Delta D_{11})^{-1}\Delta D_{12}\end{array}\right]$$

Alternatively, we can write:

$$\begin{bmatrix}A_{cl} & B_{cl}\\ C_{cl} & D_{cl}\end{bmatrix} = \bar{S}(P,\Delta) = \begin{bmatrix}A & B_2\\ C_2 & D_{22}\end{bmatrix} + \begin{bmatrix}B_1\\ D_{21}\end{bmatrix}(I-\Delta D_{11})^{-1}\Delta\begin{bmatrix}C_1 & D_{12}\end{bmatrix}$$

Linear-Fractional Representation for Matrices

There is an important point here: The LFT can be used for matrices
That is, if you have two equations:

$$\begin{bmatrix} p(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} q(t) \\ w(t) \end{bmatrix} \qquad \text{and} \qquad q(t) = \Delta p(t)$$

Then

$$z(t) = \bar{S}(M, \Delta) w(t) = \left(M_{22} + M_{21}(I - \Delta M_{11})^{-1} \Delta M_{12} \right) w(t)$$

Alternatively,

$$\begin{bmatrix} z(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ q(t) \end{bmatrix} \qquad \text{and} \qquad q(t) = \Delta p(t)$$

Becomes

$$z(t) = \underline{\mathsf{S}}(M, \Delta) w(t) = \left(M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1} M_{21} \right) w(t)$$

This works even if we replace $z(t)$ with $\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix}$ and $w(t)$ with $\begin{bmatrix} \dot{x}(t) \\ w(t) \end{bmatrix}$

Linear-Fractional Representation

Nominal System (Upper Feedback Representation):

$$\begin{bmatrix} p(t) \\ \dot{x}(t) \\ z(t) \end{bmatrix} = \left[\begin{array}{c|c} D_{11} & \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \\ \hline \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} & \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix} \end{array}\right] \begin{bmatrix} q(t) \\ x(t) \\ w(t) \end{bmatrix} = P \begin{bmatrix} q(t) \\ x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} q(t) \\ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \end{bmatrix}$$

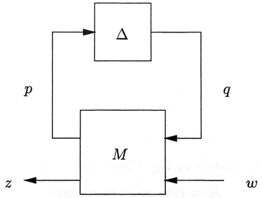
$$P_{22} = \left[\begin{smallmatrix} A & B_2 \\ C_2 & D_{22} \end{smallmatrix}\right], P_{21} = \left[\begin{smallmatrix} B_1 \\ D_{21} \end{smallmatrix}\right], P_{12} = \left[\begin{smallmatrix} C_1 & D_{12} \end{smallmatrix}\right], P_{11} = D_{11},$$

Closed-Loop: Representation of the Upper Feedback Interconnection with Δ

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \overbrace{(P_{22} + P_{21}(I - \Delta P_{11})^{-1} \Delta P_{12})}^{\bar{S}(P,\Delta)} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

Consider **Additive Uncertainty**:

$$\mathbf{P} := \{P \,:\, P = P_0 + \Delta,\, \Delta \in \mathbf{\Delta}\}$$



Nominal System: M

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & M_0 \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}$$

Uncertain System: Δ

$$q = \Delta p$$

Solving for z, we get the Upper Star Product

$$z = (M_{22} + M_{21}(I - \Delta M_{11})^{-1} \Delta M_{12}) w$$

or

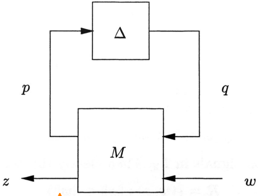
$$z = (M_0 + \Delta) w$$

Apply the LFT to Parametric Uncertainty

Multiplicative Uncertainty

Consider **Multiplicative Uncertainty**:

$$\mathbf{P} := \{P : P = (I + \Delta)P_0, \, \Delta \in \mathbf{\Delta}\}$$



Nominal System: M

$$\begin{bmatrix} p \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & M_0 \\ I & M_0 \end{bmatrix}}_{M} \begin{bmatrix} q \\ w \end{bmatrix}$$

Uncertain System: Δ

$$q = \Delta p$$

Using the Upper Star Product we get

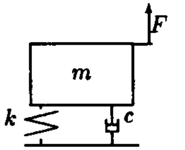
$$\bar{S}(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1} \Delta M_{12} = (I + \Delta)M_0$$

thus

$$z = (I + \Delta)M_0 w$$

Example of Parametric Uncertainty

Recall The Spring-Mass Example



$$\ddot{y}(t) = -c\dot{y}(t) - \frac{k}{m}y(t) + \frac{F(t)}{m}$$

Multiplicative Uncertainty

- $m = m_0(1 + \eta_m\delta_m)$
- $c = c_0(1 + \eta_c\delta_c)$
- $k = k_0(1 + \eta_k\delta_k)$

Define $x_1 = y$ and $x_2 = m\dot{y}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & m^{-1}\\ -k & -c\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}F$$

Nominal System Dynamics

$$\left[\begin{array}{c|c|c} A & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \\ \hline C_1 & D_{12} & D_{11}\end{array}\right] = \left[\begin{array}{cc|c|ccc} 0 & m_0^{-1} & 0 & -\eta_m & 0 & 0 \\ -k_0 & -c_0 & 1 & 0 & \eta_k & \eta_c \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & m_0^{-1} & 0 & -\eta_m & 0 & 0 \\ -k_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_0 & 0 & 0 & 0 & 0 \end{array}\right]$$

9-matrix Plant

Example of Parametric Uncertainty

Recall The Spring-Mass Example

$$\ddot{y}(t) = -c\dot{y}(t) - \frac{k}{m}y(t) + \frac{F(t)}{m}$$

Multiplicative Uncertainty

- $m = m_0(1 + \eta_m \delta_m)$
- $c = c_0(1 + \eta_c \delta_c)$
- $k = k_0(1 + \eta_k \delta_k)$

Define $x_1 = y$ and $x_2 = m\dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & m^{-1} \\ -k & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

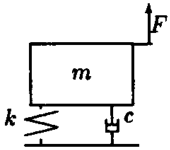
Nominal System Dynamics

$$\left[\begin{array}{c|c|c} A & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \\ \hline C_1 & D_{12} & D_{11} \end{array}\right] = \left[\begin{array}{cc|c|ccc} 0 & m_0^{-1} & 0 & -\eta_m & 0 & 0 \\ -k_0 & -c_0 & 1 & 0 & \eta_k & \eta_c \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & m_0^{-1} & 0 & -\eta_m & 0 & 0 \\ -k_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_0 & 0 & 0 & 0 & 0 \end{array}\right]$$

9-matrix Plant

Note the states $x_1 = y$ and $x_2 = m\dot{y}$ were chose carefully so as to separate the uncertain parameters.

Example of Parametric Uncertainty



Nominal System: P

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}0 & m_0^{-1}\\ -k_0 & -c_0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}F(t) + \begin{bmatrix}-\eta_m & 0 & 0\\ 0 & \eta_k & \eta_c\end{bmatrix}q(t)$$

$$p(t) = \begin{bmatrix}0 & m_0^{-1}\\ -k_0 & 0\\ 0 & -c_0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}-\eta_m & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}q(t)$$

$$z(t) = x_1(t)$$

Uncertain System: Δ

$$q = \Delta p = \begin{bmatrix}\delta_m & 0 & 0\\ 0 & \delta_k & 0\\ 0 & 0 & \delta_c\end{bmatrix} p$$

Closed-Loop:

$$\begin{bmatrix}\dot x_1(t)\\ \dot x_2(t)\\ z(t)\end{bmatrix} = (P_{22}+P_{21}(I-\Delta P_{11})^{-1}\Delta P_{12})\begin{bmatrix}x_1(t)\\ x_2(t)\\ F(t)\end{bmatrix}$$

where

$$P_{22} = \begin{bmatrix}A & B_2\\ C_2 & D_{22}\end{bmatrix}, P_{21} = \begin{bmatrix}B_1\\ D_{21}\end{bmatrix}, P_{12} = \begin{bmatrix}C_1 & D_{12}\end{bmatrix}, P_{11} = D_{11},$$

Questions:

- How to formulate the uncertainty matrix?
- What if the uncertainty is time-varying?

Formulating the LFT representation

Recall the feedback representation has the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z(t) \end{bmatrix} = (P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12}) \begin{bmatrix} x(t) \\ F(t) \end{bmatrix}$$

What types of parametric uncertainty have this form? Let

$$P_{22} = \sum_i P_{22,i}, \qquad P_{21} = \begin{bmatrix} P_{21,1} & \cdots & P_{21,1} \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} P_{12,1} \\ \vdots \\ P_{12,k} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_{11,k} \end{bmatrix} \quad \Delta = \begin{bmatrix} \delta_1 I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_k I \end{bmatrix}$$

Then

$$P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12} = \sum_i P_{22,i} + P_{21,i}(\delta_i^{-1} I - P_{11,i})^{-1} P_{12,i}$$

Hence any *Rational* Uncertainty can be represented

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z(t) \end{bmatrix} = (P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12}) \begin{bmatrix} x_1(t) \\ x_2(t) \\ w(t) \end{bmatrix}$$

In fact, ANY state-space system with rational uncertainty can be represented

Formulating the LFT representation

Recall the feedback representation has the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z(t) \end{bmatrix} = (P_{22} + P_{21}(I - \Delta P_{11})^{-1} \Delta P_{12}) \begin{bmatrix} x(t) \\ F(t) \end{bmatrix}$$

What types of parametric uncertainty have this form? Let

$$P_{22} = \sum_i P_{22,i}, \qquad P_{21} = \begin{bmatrix} P_{21,1} & \cdots & P_{21,1} \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} P_{12,1} \\ \vdots \\ P_{12,k} \end{bmatrix}, \quad P_{11} = \begin{bmatrix} P_{11,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_{11,k} \end{bmatrix} \quad \Delta = \begin{bmatrix} \delta_1 I & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \delta_k I \end{bmatrix}$$

Then

$$P_{22} + P_{21}(I - \Delta P_{11})^{-1} \Delta P_{12} = \sum_i P_{22,i} + P_{21,i}(\delta_i^{-1} I - P_{11,i})^{-1} P_{12,i}$$

Hence any *Rational* Uncertainty can be represented

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z(t) \end{bmatrix} = (P_{22} + P_{21}(I - \Delta P_{11})^{-1} \Delta P_{12}) \begin{bmatrix} x_1(t) \\ x_2(t) \\ w(t) \end{bmatrix}$$

In fact, ANY state-space system with rational uncertainty can be represented

Recall any proper, rational transfer function $\hat{G}(s)$ has a representation as

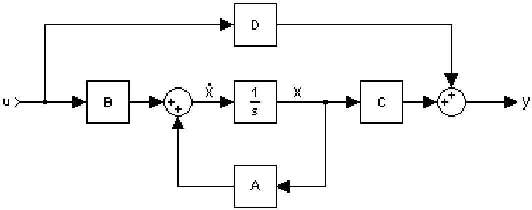
$$\hat{G}(s) = C(sI - A)^{-1} B + D$$

For each δ_i, if we can find a $\hat{G}(\delta_i^{-1})$, we can construct the corresponding LFT.

Recall:

State-Space Systems can be represented in Block-Diagram Form. e.g.

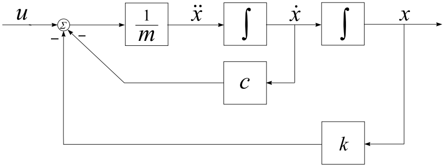
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



$$m\ddot{x} + c\dot{x} + kx = F \qquad x(s) = \frac{1}{ms^2 + cs + k}F(s)$$

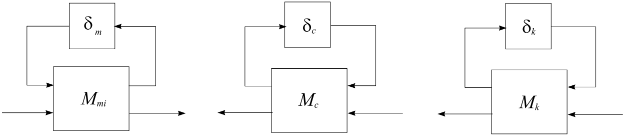
Lets consider how to do this problem in General with Block Diagrams.

Step 1: Isolate all the uncertain parameters:



Formulating the LFT

Step 2: Rewrite all the uncertain blocks as LFTs



For the $\frac{1}{m_0(1+\eta_m\delta_m)}$ Term:

$$\frac{1}{m} = \frac{1}{m_0(1+\eta_m\delta_m)} = \frac{1}{m_0} - \frac{1}{m_0}(1+\eta_m\delta_m)^{-1}\eta_m\delta_m = \bar{S}(M_m, \delta_m)$$

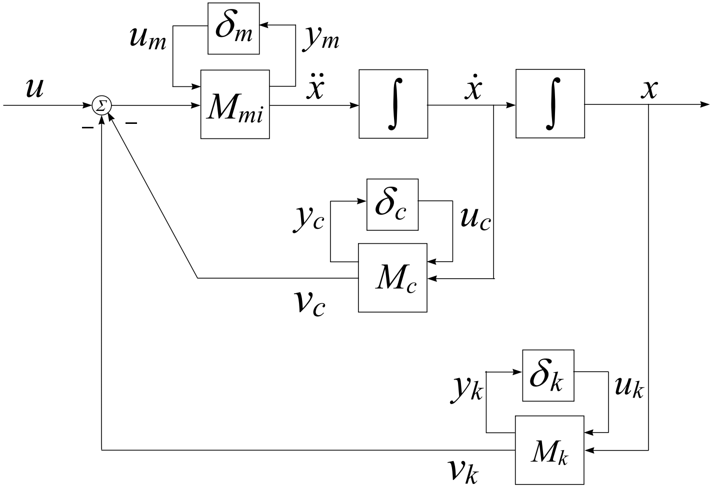
where $M_m = \begin{bmatrix} -\eta_m & \frac{1}{m_0} \\ -\eta_m & \frac{1}{m_0} \end{bmatrix}$.

For the $c_0(1+\eta_c\delta_c)$ and $k_0(1+\eta_k\delta_k)$ Terms:

$$c = c_0(1+\eta_c\delta_c) = \bar{S}(M_c, \delta_c) \qquad M_c = \begin{bmatrix} 0 & c_0 \\ \eta_c & c_0 \end{bmatrix}$$

$$k = k_0(1+\eta_k\delta_k) = \bar{S}(M_k, \delta_c) \qquad M_k = \begin{bmatrix} 0 & k_0 \\ \eta_k & k_0 \end{bmatrix}$$

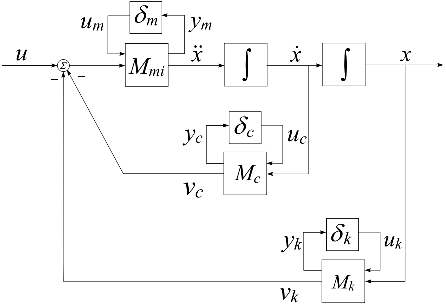
Step 3: Write down all your equations!



Set $x_1 = x$, $x_2 = \dot{x}$, $z = x_1$ so $\ddot{x} = \dot{x}_2$.

Formulating the LFT

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\eta_m u_m + \frac{1}{m_0}(w - v_c - v_k)$$
$$y_m = -\eta_m u_m + \frac{1}{m_0}(w - v_c - v_k)$$
$$y_c = c_0 x_2$$
$$y_k = k_0 x_1$$
$$v_c = \eta_c u_c + c_0 x_2, \qquad v_k = \eta_k u_k + k_0 x_1$$
$$z = x_1$$
$$u_m = \delta_m y_m, \quad u_c = \delta_c y_c, \quad u_k = \delta_k y_k$$



Eliminating v_c and v_k, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ y_m \\ y_c \\ y_k \\ z \end{bmatrix} = \left[\begin{array}{cc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & -\eta_m & -\frac{\eta_c}{m_0} & -\frac{\eta_k}{m_0} & \frac{1}{m_0} \\ \hline -\frac{k_0}{m_0} & -\frac{c_0}{m_0} & -\eta_m & -\frac{\eta_c}{m_0} & -\frac{\eta_k}{m_0} & \frac{1}{m_0} \\ 0 & c_0 & 0 & 0 & 0 & 0 \\ k_0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ u_m \\ u_c \\ u_k \\ w \end{bmatrix} \qquad u = \begin{bmatrix} \delta_m & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_c \end{bmatrix} y$$

Structured Uncertainty

In the previous example, Δ has **Structure**

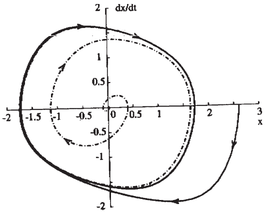
$$q = \begin{bmatrix} \delta_m & 0 & 0 \\ 0 & \delta_k & 0 \\ 0 & 0 & \delta_c \end{bmatrix} p$$

Of course, $\|\Delta\| < 1$, but it is also *diagonal*.

- To ignore this structure leads to conservative Results
- We will return to this issue in the next lecture.

The Rayleigh Equation:

$$\ddot{y} - 2\zeta(1-\alpha\dot{y}^2)\dot{y} + y = u$$



Nominal System: P

$$\dot{x}(t) = \begin{bmatrix} 2\zeta & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -2\zeta\alpha \\ 0 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$p(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

Uncertain System: Δ

$$q(t) = (\Delta p)(t) = p(t)^3$$

- Δ is NOT norm-bounded. ($p(t)^3 \not\leq Kp(t)$ for any K)
- However, $\langle p, q \rangle = \int p(t)q(t)dt = \int p(t)^4 dt \geq 0$.
- Does This Help?

Higher-Order Dynamics and Model Reduction: Missing States

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix} w$$

$$y = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + Dw$$

Problem: If we don't model the states x_2, then A_{12}, A_{21}, A_{22}, B_2 and C_2 are all unknown.

Model of Uncertainty: Put all the unknowns is an interconnected system.

Nominal System: P

$$\dot{x}_1(t) = A_{11}x_1(t) + p(t) + B_1 w(t)$$

$$q(t) = \begin{bmatrix}I\\ 0\end{bmatrix} x_1(t) + \begin{bmatrix}0\\ I\end{bmatrix} w(t)$$

Uncertain System: Δ

$$\dot{x}_2(t) = A_{22}x_1(t) + \begin{bmatrix}A_{21} & B_2\end{bmatrix} q(t)$$

$$p(t) = A_{12}x_2(t)$$

Question: How to model $\boldsymbol{\Delta}$ if it is unknown?

- Since Δ is state-space (and stable), $\Delta \in H_\infty$.
- Which means $\|\Delta\|_{\mathcal{L}(L_2)} = \|\Delta\|_{H_\infty}$ is bounded.
- Can we assume $\|\Delta\|_{H_\infty} < 1$? $< .1$?

Time-Varying Uncertainty?

Gain Scheduling and Logical Switching

Several Operating Points:

Table 11.2 Parameter Values at the Seven Operating Points

Time (s)	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$a_1(t)$	1.593	1.485	1.269	1.130	0.896	0.559	0.398
$a_1'(t)$	0.285	0.192	0.147	0.118	0.069	0.055	0.043
$a_2(t)$	260.559	266.415	196.737	137.385	129.201	66.338	51.003
$a_3(t)$	185.488	182.532	176.932	160.894	138.591	78.404	53.840
$a_4(t)$	1.506	1.295	1.169	1.130	1.061	0.599	0.421
$a_5(t)$	0.298	0.243	0.217	0.191	0.165	0.105	0.078
$b_1(t)$	1.655	1.502	1.269	1.130	0.896	0.559	0.398
$b_1'(t)$	0.295	0.195	0.147	0.118	0.069	0.055	0.043
$b_2(t)$	39.988	−24.627	−31.452	−41.425	−68.165	−21.448	−9.635
$b_3(t)$	159.974	170.532	182.030	184.093	154.608	89.853	59.587
$b_4(t)$	0.771	0.652	0.680	0.691	0.709	0.360	0.243
$b_5(t)$	0.254	0.191	0.188	0.182	0.162	0.102	0.072

Dynamics:

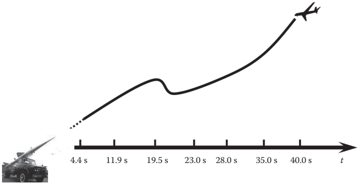
$$\dot{x}(t) = Ax(t) + Bu(t)$$

`If x(t) <3 :` $u(t) = K_1 x(t)$

`If x(t) >3 :` $u(t) = K_2 x(t)$

There can be an array of gains.

In **Gain Scheduling**, the controller switches depending on operating point.



The dynamics switch with the state.

- This is called a Hybrid System
- Technically, it is not uncertain, since model is defined

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bu(t)$$

Nominal System: P

$$\dot{x}(t) = Ax(t) + Aq(t) + Bu(t)$$
$$p(t) = x(t)$$
$$y(t) = x(t)$$

Uncertain System: Δ

$$q(t) = p(t-\tau)$$

In the Frequency Domain:

$$q(s) = e^{-\tau s} p(s)$$

Hence $\hat{\Delta}(s) = e^{-\tau s}$

- $\|\hat{\Delta}\|_{H_\infty} = 1$
- Can use Small-gain.

Alternatives to the LFT

Additive Affine Time-Varying Interval and Polytopic Uncertainty

- Time-Varying Uncertainty can cause problems
- Because dealing with *Structured Uncertainty* is difficult, we often look for alternative representations.

Consider the following form of time-varying uncertainty

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

where

$$\Delta A(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$

where $\delta(t)$ lies in either the intervals

$$\delta_i(t) \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta(t) \in \{\alpha \,:\, \sum_i \alpha_i = 1,\, \alpha_i \geq 0\}$$

For convenience, we denote this *Convex Hull* as

$$Co(A_1, \cdots, A_k) := \left\{ \sum_i A_i\alpha_i \;:\; \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$

Alternatives to the LFT

Additive Affine Time-Varying Interval and Polytopic Uncertainty

For example,

$$m\ddot{x} + c\dot{x} + kx = F \qquad x(s) = \frac{1}{ms^2 + cs + k}u(s)$$

Define $x_1 = y$ and $x_2 = m\dot{y}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & m^{-1}\\ -k & -\frac{c}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}F$$

Then if $m \in [m^-, m^+]$, $c \in [c^-, c^+]$, $k \in [k^-, k^+]$, then

$$m^{-1} \in \left[\frac{1}{m^+}, \frac{1}{m^-}\right]$$

$$\frac{c}{m} \in \left[\frac{c^-}{m^+}, \frac{c^+}{m^-}\right]$$

Note: This doesn't always work!

- e.g. if in addition there were a c coefficient (appearing w/o $1/m$).
- Need a change of parameters which becomes affine in the parameters.
- Then you are stuck with the LFT.

Discrete-Time Case

All frameworks are readily adapted to the Discrete-Time Case:
LFT Framework:

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \bar{S}(P, \Delta) \begin{bmatrix} x_k \\ w_k \end{bmatrix}$$

Additive or Polytopic Framework:

$$x_{k+1} = (A_0 + \Delta A_k)x_k + (B_0 + \Delta B_k)u_k$$

where

$$\Delta A_k = A_1 \delta_{1,k} + \cdots + A_k \delta_{K,k}$$

where δ_k lies in either the intervals

$$\delta_{i,k} \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta_k \in \{\alpha \,:\, \sum_i \alpha_i = 1,\ \alpha_i \geq 0\}$$

Types of Uncertainty

To Summarize, we have many choices for our uncertainty Set, $\mathbf{\Delta}$

- **Unstructured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$

- **Structured, Static, norm-bounded:**

$$\mathbf{\Delta} := \{\text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) \,:\, |\delta_i| < 1,\ \bar{\sigma}(\Delta_i) < 1\}$$

- **Structured, Dynamic, norm-bounded:**

$$\mathbf{\Delta} := \{\text{diag}(\Delta_1, \Delta_2, \cdots) \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$

- **Unstructured, Parametric, norm-bounded:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

- **Parametric, Polytopic:**

$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\ \alpha_i \ge 0,\ \sum_i \alpha_i = 1\}$$

- **Parametric, Interval:**

$$\mathbf{\Delta} := \left\{\sum_i \Delta_i \delta_i \,:\, \delta_i \in [\delta_i^-, \delta_i^+]\right\}$$

Each of these can be Time-Varying or Time-Invariant!