LMI Methods in Optimal and Robust Control

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Thanks to S. Lall and P. Parrilo for guidance and supporting material

Lecture 16: Optimization of Polynomials and an LMI for Global Lyapunov Stability

Optimization of Polynomials:

As Opposed to Polynomial Programming

Polynomial Programming (NOT CONVEX): n decision variables

$$\min_{x\in\mathbb{R}^n} \quad f(x)$$
$$g_i(x) \geq 0$$

- f and g_i must be convex for the problem to be convex.

Optimization of Polynomials: Lifting to a higher-dimensional space

$$\begin{aligned} &\max_{g,\gamma} \quad \gamma \\ &f(x) - \gamma = h(x) \qquad \text{for all} \quad x \in \mathbb{R}^n \\ &h(x) \geq 0 \qquad \text{for all} \quad x \in \{x \in \mathbb{R}^n \ : \ g_i(x) \geq 0\} \end{aligned}$$

- The decision variables are *functions* (e.g. g)
  - One constraint for *every possible value of* x.
- But how to parameterize functions????
- How to enforce an infinite number of constraints???
- Advantage: Problem is convex, even if f, g, h are not convex.

Of course, you already know some applications of Optimization of Polynomials

- Global Stability of Nonlinear Systems

$$V(x) > \epsilon x^2 \qquad \text{for all } x \in \mathbb{R}^n$$
$$\nabla V(x)^T f(x) < 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Stability of Systems with Positive States: Not all states can be negative...

- Cell Populations/Concentrations
- Volume/Mass/Length

We want:

$$V(x) = x^T P x \geq 0 \qquad \text{for all} \qquad x \geq 0$$
$$\dot{V}(x) = x^T (A^T P + PA) x \leq 0 \qquad \text{for all} \qquad x \geq 0$$

- Matrix Copositivity (An NP-hard Problem)

Verify:

$$x^T P x \geq 0 \qquad \text{for all} \quad x \geq 0$$

Recall: Systems with Uncertainty

$$\begin{aligned}\dot{x}(t) &= A(\delta)x(t) + B_1(\delta)w(t) + B_2(\delta)u(t)\\ y(t) &= C(\delta)x(t) + D_{12}(\delta)u(t) + D_{11}(\delta)w(t)\end{aligned}$$

Theorem 1.

There exists an $F(\delta)$ *such that* $\|\underline{S}(P(\delta), K(0,0,0,F(\delta)))\|_{H_\infty} \le \gamma$ *for all* $\delta \in \mathbf{\Delta}$ *if there exist* $Y > 0$ *and* $Z(\delta)$ *such that*

$$\begin{bmatrix} YA(\delta)^T + A(\delta)Y + Z(\delta)^T B_2(\delta)^T + B_2(\delta)Z(\delta) & *^T & *^T \\ B_1(\delta)^T & -\gamma I & *^T \\ C_1(\delta)Y + D_{12}(\delta)Z(\delta) & D_{11}(\delta) & -\gamma I \end{bmatrix} < 0 \quad \text{for all} \quad \delta \in \mathbf{\Delta}$$

Then $F(\delta) = Z(\delta)Y^{-1}$.

The Structured Singular Value, μ

Definition 2.

Given system $M \in \mathcal{L}(L_2)$ and set $\mathbf{\Delta}$ as above, we define the **Structured Singular Value** of $(M, \mathbf{\Delta})$ as

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf_{\substack{\Delta \in \mathbf{\Delta} \\ I - M\Delta \text{ is singular}}} \|\Delta\|}$$

The system

$$\dot{x}(t) = A_0 x(t) + M q(t), \qquad q(t) = \Delta(t) p(t),$$
$$p(t) = N x(t) + Q q(t), \qquad \Delta \in \mathbf{\Delta}$$

Lower Bound for μ: $\mu \geq \gamma$ if there exists a $P(\delta)$ such that

$$P(\delta) \geq 0 \qquad \text{for all} \quad \delta$$

$$\dot{V} = x^T P(\delta)(A_0 x + M q) + (A_0 x + M q)^T P(\delta) x < \epsilon x^T x$$

for all x, q, δ such that

$$(x, q, \delta) \in \left\{ x, q, \delta \,:\, q = \text{diag}(\delta_i)(N x + Q q),\, \sum_i \delta_i^2 \leq \gamma^2 \right\}$$

Overview

In this lecture, we will show how the LMI framework can be expanded dramatically to other forms of control problems.

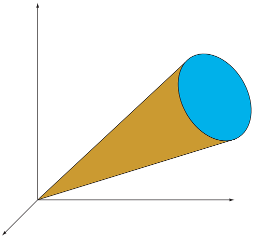
1. Positivity of Polynomials
   1.1 Sum-of-Squares
2. Positivity of Polynomials on Semialgebraic sets
   2.1 Inference and Cones
   2.2 Positivstellensatz
3. Applications
   3.1 Nonlinear Analysis
   3.2 Robust Analysis and Synthesis
   3.3 Global optimization

Is Optimization of Polynomials Tractable or Intractable?

The Answer lies in Convex Optimization

A Generic Convex Optimization Problem:

$$\max_x \; bx$$
$$\text{subject to} \quad Ax \in C$$



The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Computational Tractability: Convex Optimization over C is tractable if

- The set membership test for $y \in C$ is in P (polynomial-time verifiable).
- The variable x is a finite dimensional vector (e.g. $\mathbb{R}^n$).

Optimization of Polynomials is Convex

The variables are finite-dimensional (if we bound the degree)

Convex Optimization of Functions: Variables $V \in \mathcal{C}[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \; \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

V is the decision variable (infinite-dimensional)

- How to make it finite-dimensional???

The set of polynomials is an infinite-dimensional (but *Countable*) vector space.

- It is **Finite Dimensional** if we bound the degree
- All finite-dimensional vector spaces are equivalent!

But we need a way to parameterize this space...

To Begin: How do we Parameterize Polynomials???

A Parametrization consists of a **basis** and a **set of parameters** (coordinates)

- The set of polynomials is an infinite-dimensional vector space.
- It is **Finite Dimensional** if we bound the degree
  - The monomials are a simple basis for the space of polynomials

Definition 3.

Define $Z_d(x)$ to be the vector of monomial bases of degree d or less.

e.g., if $x \in \mathbb{R}^2$, then the vector of basis functions is

$$Z_2(x_1, x_2)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

and

$$Z_4(x_1)^T = \begin{bmatrix} 1 & x_1 & x_1^2 & x_2^3 & x_1^4 \end{bmatrix}$$

Linear Representation

- Any polynomial of degree d can be represented with a vector $c \in \mathbb{R}^m$

$$p(x) = c^T Z_d(x)$$

- c is the vector of *parameters* (decision variables).
- $Z_d(x)$ doesn't change (fixed).

$$2x_1^2 + 6x_1x_2 + 4x_2 + 1 = \begin{bmatrix} 1 & 0 & 4 & 6 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}^T$$

Optimization of Polynomials is Convex

The variables are finite-dimensional (if we bound the degree)

Convex Optimization of Functions: Variables $V \in \mathbb{R}[x]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

Now use the polynomial parametrization $V(x) = c^T Z(x)$

- Now c is the decision variable.

Convex Optimization of Polynomials: Variables $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$

$$\max_{c,\gamma} \gamma$$

subject to

$$c^T Z(x) - x^T x \geq 0 \qquad \forall x$$
$$c^T \nabla Z(x) f(x) + \gamma x^T x \leq 0 \qquad \forall x$$

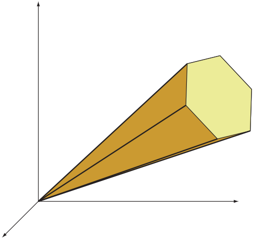
Can LMIs be used for Optimization of Polynomials???

Optimization of Polynomials is NP-Hard!!!

Problem: Use a finite number of variables:

$$\max\ b^T x$$

$$\text{subject to } A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$$



The A_i are matrices of polynomials in y. e.g. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha}\ y^\alpha$$

The FEASIBLITY TEST is Computationally Intractable

The problem: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" (i.e. "$p \in \mathbb{R}^+[x]$?") is NP-hard.

How to Find Lyapunov Functions (LF)?

We know a LF by its Properties

What makes a LF a Lyapunov Function?

Property 1: Positivity

- The Lyapunov function must be positive (metrics are positive).
- Also $V(0) = 0$ if 0 is an equilibrium.

Property 2: Negativity along Trajectories

- A Lyapunov Function decreases monotonically in time.
  - A longer trajectory is always "bigger" than a shorter one.
  - As time progresses, the trajectory gets shorter.
  - Hence the LF is always decreasing.

Thus: If a function has properties 1) and 2), it is a Lyapunov Function

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Note: For Linear Systems, we can restrict the search to quadratic LFs.

Property 3: Quadratic (e.g. $x^T P x$)

- If the system is Linear, the solution map is Linear
- Then if the metric is Quadratic, the Lyapunov function is Quadratic.
  - The Composition of Linear and Quadratic Functions is Quadratic

Lyapunov Functions for Linear ODEs

$$\dot{x}(t) = Ax(t)$$

if $x \in \mathbb{R}^n$, then any quadratic function has the form:

$$V(x)_{[1]} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}^T}_{x^T} \underbrace{\begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} & P_{51} \\ P_{21} & P_{22} & P_{32} & P_{42} & P_{52} \\ P_{31} & P_{32} & P_{33} & P_{43} & P_{53} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{54} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix}}_{P \geq 0} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} \geq 0$$

P3 - Quadratic: The 15 *variables* P_{ij} parameterize all possible quadratic functions on $x \in \mathbb{R}^5$.
P1 - Positive: $V(x) > 0$ for all $x \neq 0$ if and only if P has all positive eigenvalues.

Lyapunov Functions for Nonlinear ODEs

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = f(x(t), y(t))$$

Lets try a quadratic function of the form:

$$V(x,y)_{[2]} = \underbrace{\begin{bmatrix} x \\ y \\ xy \\ x^2 \\ y^2 \end{bmatrix}^T}_{Z(x,y)^T} \underbrace{\begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} & P_{51} \\ P_{21} & P_{22} & P_{32} & P_{42} & P_{52} \\ P_{31} & P_{32} & P_{33} & P_{43} & P_{53} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{54} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} \end{bmatrix}}_{P \geq 0} \underbrace{\begin{bmatrix} x \\ y \\ xy \\ x^2 \\ y^2 \end{bmatrix}}_{Z(x,y)} \geq 0$$

P3 - Pseudo-Quadratic: The 15 *variables* P_{ij} parameterize positive polynomial functions $x, y \in \mathbb{R}^2$ of degree $d \leq 4$.
P1 - Positive: $V(x) > 0$ for all $x \neq 0$ if P has all positive eigenvalues.

Can LMIs be used to Optimize Positive Polynomials???

Show Me the LMI!

Basic Idea: If there exists a Positive Matrix $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

Positive Matrices ($P \geq 0$) have square roots!

$$P = Q^T Q$$

Hence

$$\begin{aligned} V(x) &= Z_d(x)^T Q^T Q Z_d(x) = (Q Z_d(x))^T (Q Z_d(x)) \\ &= h(x)^T h(x) \geq 0 \end{aligned}$$

Conclusion:

$$V(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n$$

if there exists a $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

- Such a function is called Sum-of-Squares (SOS), denoted $V \in \Sigma_s$.
- This is an LMI! Equality constraints relate the coefficients of V (decision variables) to the elements of P (more decision variables).

How Hard is it to Determine Positivity of a Polynomial???

Definition 4.

A Polynomial, f, is called Positive SemiDefinite (PSD) if

$$f(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

The Primary Problem: How to enforce the constraint $f(x) \geq 0$ for all x?

Easy Proof: Certificate of Infeasibility

- A Proof that f is NOT PSD.
- i.e. To show that

$$f(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

  is FALSE, we need only find a point x with $f(x) < 0$.

Complicated Proof: It is much harder to identify a **Certificate of Feasibility**

- A Proof that f is PSD.

Global Positivity Certificates (Proofs and Counterexamples)

Question: How does one prove that $f(x)$ is positive semidefinite?

What Kind of Functions do we Know are PSD?

- Any squared function is positive.
- The sum of squared forms is PSD
- The product of squared forms is PSD
- The ratio of squared forms is PSD

So $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ if

$$V(x) = \prod_k \frac{\sum_i f_{ik}(x)^2}{\sum_j h_{jk}(x)^2}.$$

But is any PSD polynomial the sum, product, or ratio of squared polynomials?

- An old Question....

Definition 5.

A polynomial, $p(x) \in \mathbb{R}[x]$ is a **Sum-of-Squares (SOS)**, denoted $p \in \Sigma_s$ if there exist polynomials $g_i(x) \in \mathbb{R}[x]$ such that

$$p(x) = \sum_{i=1}^{k} g_i(x)^2.$$

David Hilbert created a famous list of 23 then-unsolved mathematical problems in 1900.

- Only 10 have been fully resolved.
- The 17th problem has been resolved.

  "Given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions?" *-D. Hilbert, 1900*

Hilbert's 17th was resolved in the affirmative by E. Artin in 1927.

- Any PSD polynomial is the sum, product and ratio of squared polynomials.
- If $p(x) \geq 0$ for all $x \in \mathbb{R}^n$, then

$$p(x) = \frac{g(x)}{h(x)}$$

  where $g, h \in \Sigma_s$.
- If p is positive definite, then we can assume $h(x) = (\sum_i x_i^2)^d$ for some d. That is,

$$(x_1^2 + \cdots + x_n^2)^d p(x) \in \Sigma_s$$

- If we can't find a SOS representation (certificate) for $p(x)$, we can try $(\sum_i x_i^2)^d p(x)$ for higher powers of d.

Of course this doesn't answer the question of how we find SOS representations.

Quadratic Parameterization of Polynomials

Quadratic Representation

- Alternative to Linear Parametrization, a polynomial of degree d can be represented by a matrix $M \in \mathbb{S}^m$ as
$$p(x) = Z_d(x)^T M Z_d(x)$$
- However, now the problem may be under-determined
$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$
$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 x y^3 + M_6 y^4$$

Thus, there are infinitely many quadratic representations of p. For the polynomial
$$f(x) = 4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4,$$
we can use the alternative solution
$$4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4$$
$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 x y^3 + M_6 y^4$$

Polynomial Representation - Quadratic

For the polynomial

$$f(x) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4,$$

we require

$$\begin{aligned} &4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \\ &\qquad = M_1x^4 + 2M_2x^3y + (2M_3 + M_4)x^2y^2 + 2M_5xy^3 + M_6y^4 \end{aligned}$$

Constraint Format:

$$M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad 10 = M_6.$$

An underdetermined system of linear equations (6 variables, 5 equations).

- This yields a family of quadratic representations, parameterized by λ as

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

  which holds for any $\lambda \in \mathbb{R}$

Positive Matrix Representation of SOS

Sufficiency

Quadratic Form:

$$p(x) = Z_d(x)^T M Z_d(x)$$

Consider the case where the matrix M is positive semidefinite.

Suppose: $p(x) = Z_d(x)^T M Z_d(x)$ where $M > 0$.

- Any positive semidefinite matrix, $M \geq 0$ has a square root $M = PP^T$

Hence

$$p(x) = Z_d(x)^T M Z_d(x) = Z_d(x)^T P P^T Z_d(x).$$

Which yields

$$p(x) = \sum_i \left(\sum_j P_{i,j} Z_{d,j}(x) \right)^2$$

which makes $p \in \Sigma_s$ an SOS polynomial.

Moreover: Any SOS polynomial has a quadratic rep. with a PSD matrix.

Suppose: $p(x) = \sum_i g_i(x)^2$ is degree $2d$ (g_i are degree d).

- Each $g_i(x)$ has a linear representation in the monomials.
$$g_i(x) = c_i^T Z_d(x)$$
- Hence
$$p(x) = \sum_i g_i(x)^2 = \sum_i Z_d(x) c_i c_i^T Z_d(x) = Z_d(x) \left(\sum_i c_i c_i^T \right) Z_d(x)$$
- Each matrix $c_i c_i^T \geq 0$. Hence $Q = \sum_i c_i c_i^T \geq 0$.
- We conclude that if $p \in \Sigma_s$, there is a $Q \geq 0$ with $p(x) = Z_d(x) Q Z_d(x)$.

Lemma 6.

Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ if and only if there exists some $Q \succeq 0$ such that
$$M(x) = Z_d(x)^T Q Z_d(x).$$

Sum-of-Squares

Thus we can express the search for a SOS certificate of positivity as an LMI.

Take the numerical example

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

The question of an SOS representation is equivalent to

$$\text{Find} \quad M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \geq 0 \quad \text{such that}$$

$$M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad M_6 = 10.$$

In fact, this is feasible for

$$M = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

Sum-of-Squares

We can use this solution to construct an SOS certificate of positivity.

$$
\begin{aligned}
4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 &= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \\
&= \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix}^T \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix} \\
&= (2xy + y^2)^2 + (2x^2 + xy + 3y^2)^2
\end{aligned}
$$

Quadratic Representation: (Using Matrix $M \in \mathbb{R}^{p \times p}$):

$$p(x) = Z_d(x)^T M Z_d(x)$$

Linear Representation: (Using Vector $c \in \mathbb{R}^q$)

$$q(x) = c^T Z_{2d}(x)$$

To constrain $p(x) = q(x)$, we write $[Z_d]_i = x^{\alpha_i}$, $[Z_{2d}]_j = x^{\beta_j}$ and reformulate

$$p(x) = Z_d(x)^T M Z_d(x) = \sum_{i,j} M_{i,j} x^{\alpha_i + \alpha_j} = \mathsf{vec}(M)^T A Z_{2d}(x)$$

where $A \in \mathbb{R}^{p^2 \times q}$ is defined as

$$A_{i,j} = \begin{cases} 1 & \text{if } \alpha_{\mathsf{mod}(i,p)} + \alpha_{\lfloor i \rfloor_p + 1} = \beta_j \\ 0 & \text{otherwise} \end{cases}$$

This then implies that

$$Z_d(x)^T M Z_d(x) = \mathsf{vec}(M)^T A Z_{2d}(x)$$

Hence if we constrain $c = \mathsf{vec}(M)^T A$, this is equivalent to $p(x) = q(x)$

Solving Sum-of-Squares using SDP

Quadratic vs. Linear Representation

Summarizing, e.g., for Lyapunov stability, we have variables $M > 0, Q > 0$ with the constraint

$$-\mathsf{vec}(M)^T A = \mathsf{vec}(Q)^T AB$$

Feasibility implies stability since

$$\begin{aligned} V(x) &= Z(x)^T Q Z(x) \geq 0 \\ \dot{V}(x) &= \mathsf{vec}(Q)^T A \nabla Z_{2d}(x) \\ &= \mathsf{vec}(Q)^T AB Z_{2d}(x) \\ &= -\mathsf{vec}(M)^T A Z_{2d}(x) \\ &= -Z(x)^T M Z(x) \geq 0 \end{aligned}$$

YALMIP has SOS functionality
Link: YALMIP SOS Manual

To test whether

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

is a positive polynomial, we use:

> sdpvar x y
> $p = 4 * x^4 + 4 * x^3 * y - 7 * x^2 * y^2 - 2 * x * y^3 + 10 * y^4;$
> F=[];
> F=[F;sos(p)];
> solvesos(F);

To retrieve the SOS decomposition, we use

> sdisplay(p)

> $ans =$
> $'1.7960 * x^2 - 3.0699 * y^2 + 0.6468 * x * y'$
> $' - 0.6961 * x^2 - 0.7208 * y^2 - 1.4882 * x * y'$
> $'0.5383 * x^2 + 0.2377 * y^2 - 0.3669 * x * y'$

In this class, we will use instead SOSTOOLS
Link: SOSTOOLS Website

To test whether

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

is a positive polynomial, we use:

```
> pvar x y
>  p = 4*x^4 + 4*x^3*y - 7*x^2*y^2 - 2*x*y^3 + 10*y^4;
> prog=sosprogram([x y]);
> prog=sosineq(prog,p);
> prog=sossolve(prog);
```

This also works for matrix-valued polynomials.

$$M(y,z)=\begin{bmatrix}(y^2+1)z^2 & yz\\ yz & y^4+y^2-2y+1\end{bmatrix}$$

$$\begin{bmatrix}(y^2+1)z^2 & yz\\ yz & y^4+y^2-2y+1\end{bmatrix}=\begin{bmatrix}z & 0\\ yz & 0\\ 0 & 1\\ 0 & y\\ 0 & y^2\end{bmatrix}^T\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 1 & -1 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & -1 & -1 & 1 & 0\\ 1 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}z & 0\\ yz & 0\\ 0 & 1\\ 0 & y\\ 0 & y^2\end{bmatrix}$$

$$=\begin{bmatrix}z & 0\\ yz & 0\\ 0 & 1\\ 0 & y\\ 0 & y^2\end{bmatrix}^T\begin{bmatrix}0 & 1 & 1 & -1 & 0\\ 1 & 0 & 0 & 0 & 1\end{bmatrix}^T\begin{bmatrix}0 & 1 & 1 & -1 & 0\\ 1 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}z & 0\\ yz & 0\\ 0 & 1\\ 0 & y\\ 0 & y^2\end{bmatrix}$$

$$=\begin{bmatrix}yz & 1-y\\ z & y^2\end{bmatrix}^T\begin{bmatrix}yz & 1-y\\ z & y^2\end{bmatrix}\in\Sigma_s$$

This also works for matrix-valued polynomials.

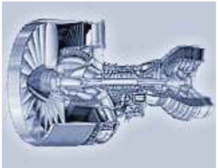
$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

SOSTOOLS Code: Matrix Positivity

```
> pvar x y
> M = [(y^2 + 1) * z^2  y * z; y * z  y^4 + y^2 - 2 * y + 1];
> prog=sosprogram([y z]);
> prog=sosmatrixineq(prog,M);
> prog=sossolve(prog);
```

An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).



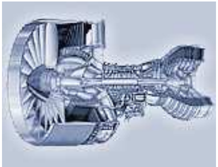
$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$

SOSTOOLS Code: Global Stability

```
> pvar x y
> f = [-y - 1.5*x^2 - .5*x^3; 3*x - y];
> prog=sosprogram([x y]);
> Z=monomials([x,y],0:2);
> [prog,V]=sossosvar(prog,Z);
> V = V + .0001*(x^4 + y^4);
> prog=soseq(prog,subs(V,[x; y],[0; 0]));
> nablaV=[diff(V,x);diff(V,y)];
> prog=sosineq(prog,-nablaV'*f);
> prog=sossolve(prog);
> Vn=sosgetsol(prog,V)
```

Finds a Lyapunov Function of degree 4.

An Example of Global Stability Analysis



A controlled model of a jet engine (Derived from Moore-Greitzer).

$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$

$$\dot{y} = 3x - y$$

YALMIP Code: Global Stability

```
> sdpvar x y
> f = [-y - 1.5*x^2 - .5*x^3; 3*x - y];
> [V,Vc]=polynomial([x y],4);
> F=[Vc(1)==0];
> F = [F;sos(V - .00001*(x^2 + y^2))];
> nablaV=jacobian(V,[x y]);
> F=[F;sos(-nablaV*f)];
> solvesos(F,[],[],[Vc])
```

Finds a Lyapunov Function of degree 4.

- Going forward, we will use mostly SOSTOOLS

SOSOPT and DelayTOOLS

There is a third relatively new Parser called SOSOPT

Link: SOSOPT Website

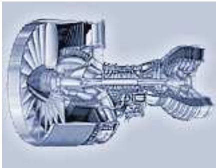
And I can plug my own mini-toolbox version of SOSTOOLS:

Link: DelayTOOLS Website

- However, I don't expect you to need this toolbox for this class.

An Example of Global Stability Analysis

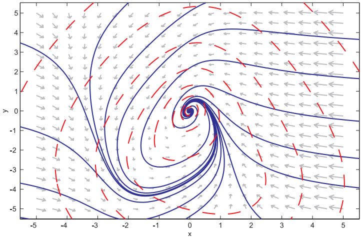
A controlled model of a jet engine (Derived from Moore-Greitzer).



$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$

This is feasible with

$$V(x) = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2$$
$$+ 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.090723y^4$$



Summary of the SOS Conditions

Proposition 1.

Suppose: $p(x) = Z_d(x)^T Q Z_d(x)$ *for some* $Q > 0$. *Then* $p(x) \geq 0$ *for all* $x \in \mathbb{R}^n$

Refinement 1: Suppose $Z_d(x)^T P Z_d(x) p(x) = Z_d(x)^T Q Z_d(x)$ for some $Q, P > 0$. Then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Refinement 2: Suppose $(\sum_i x_i^2)^q p(x) = Z_d(x)^T Q Z_d(x)$ for some $P > 0$, $q \in \mathbb{N}$. Then $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Ignore these Refinements

- SOS by itself is sufficient. The refinements are Necessary and Sufficient.
- Almost never necessary in practice...

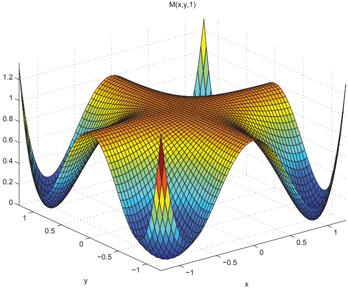
Problems with SOS

Unfortunately, a Sum-of-Squares representation is not necessary for positivity.

- Artin included ratios of squares.

Counterexample: The Motzkin Polynomial

$$M(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$



However, $(x^2 + y^2 + 1)M(x,y)$ is a Sum-of-Squares.

$$\begin{aligned}(x^2 + y^2 + 1)M(x,y) &= (x^2y - y)^2 + (xy^3 - x)^2 + (x^2y^2 - 1)^2 \\ &\quad + \frac{1}{4}(xy^3 - x^3y)^2 + \frac{3}{4}(xy^3 + x^3y - 2xy)^2\end{aligned}$$