

LMI Methods in Optimal and Robust Control

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Lecture 03: Relaxations, Duality, Cones, Positive Matrices, LMIs

What is Optimization?

An Optimization Problem has 3 parts.

$$\begin{aligned} \min_{x \in \mathbb{F}} \quad & f(x) : && \text{subject to} \\ & g_i(x) \leq 0 && i = 1, \dots, K_1 \\ & h_i(x) = 0 && i = 1, \dots, K_2 \end{aligned}$$

Variables: $x \in \mathbb{F}$

- The things you must choose.
- Typically vectors or matrices.

Objective: $f(x)$

- A function which assigns a *scalar* value to any choice of variables.

Constraints: $g(x) \leq 0; h(x) = 0$

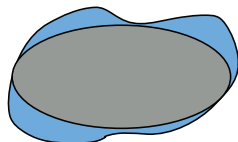
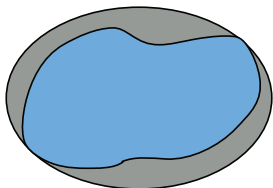
- Defines what is a minimally acceptable choice of variables.
- Equality and Inequality constraints are common.

New Concept: Relaxations and Tightenings

How to Approximate a Non-Convex Problem (Using a Convex Approximation)

Original Problem:

$$\gamma^* := \min_{x \in \mathbb{R}} f(x) : \quad g(x) \geq 0 \text{ (FS)}$$



Definition 1.

In a **Relaxation**, we remove or loosen one of the constraints.

$$\gamma_R^* := \min_{x \in \mathbb{R}} f(x) : \quad g(x) \geq -1$$

- $\gamma_R^* \leq \gamma^*$
- Solution x^* no longer feasible.
- An *Outer Approximation* of FS.

Definition 2.

In a **Tightening**, we add new constraints.

$$\gamma_T^* := \min_{x \in \mathbb{R}} f(x) : \quad g(x) \geq 1$$

- $\gamma_T^* \geq \gamma^*$
- Solution x^* is still feasible.
- An *Inner Approximation* of FS.

New Concept: Relaxations and Tightenings

Original Problem: $\gamma^* := \min_{x \in \mathcal{C}} f(x) \quad g(x) \geq 0$ (FS)**Definition 1.**

In a Relaxation, we remove or loosen one of the constraints.

$$\bar{\gamma}_R := \min_{x \in \bar{\mathcal{C}}} f(x) \quad g(x) \geq -1$$

- $\bar{\gamma}_R \leq \gamma^*$
- Solution x^* no longer feasible.
- An Outer Approximation of FS.

Definition 2.

In a Tightening, we add new constraints.

$$\bar{\gamma}_T := \min_{x \in \bar{\mathcal{C}}} f(x) \quad g(x) \geq 1$$

- $\bar{\gamma}_T \geq \gamma^*$
- Solution x^* is still feasible.
- An Inner Approximation of FS.

- FS stands for Feasible Set
 - The set of values of x which satisfy the constraints
- Relaxations *Increase* the size of the feasible set
 - The solution may not be feasible for the original problem
- Tightenings *Decrease* the size of the feasible set
 - The solution may not be optimal for the original problem

Relaxations and Tightenings

Examples

MAX-CUT: Original Problem

$$\max_{x_i^2=1} \frac{1}{2} \sum_{i,j} w_{i,j} (1 - x_i x_j)$$

Solution: $\gamma^* = 4$

- $x_1 = x_3 = x_4 = 1$
- $x_2 = x_5 = -1$

YALMIP Code:

```
> x = sdpvar(5,1);  
> F=[-1 <= x <= 1];  
> obj=2.5-.5*x(1)*x(2)-.5*x(2)*x(3)  
>      -.5*x(3)*x(4)-.5*x(4)*x(5)-.5*x(1)*x(5);  
> optimize(F,-obj);  
> value(x);
```

Link: [Download the Free version of IBM solver CPLEX and add path to MATLAB](#)

MAX-CUT: Relaxed Problem

$$\max_{x_i^2 \leq 1} \frac{1}{2} \sum_{i,j} w_{i,j} (1 - x_i x_j)$$

Solution: $\gamma_R^* = 4$

- $x_2 = x_5 = 1$
- $x_1 = x_4 = -1$
- $x_3 = 0$

MAX-CUT: Original Problem

$$\max_{x_i \in \{0,1\}} \sum_{i,j} w_{i,j}(1 - x_i x_j)$$

Solution: $\gamma_1^* = 4$

- $x_1 = x_2 = x_3 = 1$
- $x_4 = x_5 = -1$

MAX-CUT: Relaxed Problem

$$\max_{x_i \in [-1,1]} \sum_{i,j} w_{i,j}(1 - x_i x_j)$$

Solution: $\gamma_1^* = 4$

- $x_1 = x_3 = 1$
- $x_2 = x_4 = -1$
- $x_5 = 0$

YALMIP Code:

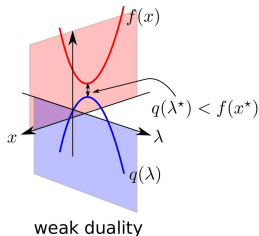
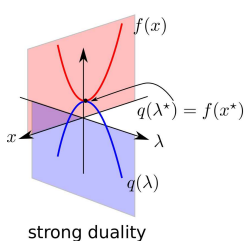
```
> z = sdpvar(5,1);
> S=[1 0 0 0 1];
> obj=2.5*.5*x(1)*x(2)+.5*x(2)*x(3)
> - .5*x(3)*x(4)+.5*x(4)*x(5)+.5*x(1)*x(5);
> optimize(P,obj);
> value(x);
```

Link: [Download the Free version of IBM solver CPLEX and add path to MATLAB](#)

Note the solution to the relaxed Max Cut problem is not feasible for the original problem.

A Third Option: Duality

A Cool Word, but Meaning is Vague



Definition 3.

Two **Optimization Problems are Dual** if any feasible solution to one has objective value which bounds the solution to the other problem.

Primal Problem:

$$\min_{x \in \mathbb{R}} f(x) : \quad x \in S$$

Dual Problem:

$$\max_{y \in \mathbb{R}} f_D(y) : \quad y \in S_D$$

Relationship:

- if $y \in S_D$, then $f_D(y) \leq f(x)$ for any $x \in S$.
- if $x \in S$, then $f(x) \geq f_D(y)$ for any $y \in S_D$.

Lagrangian Duality

$$\min_{x \in \mathbb{F}} f_0(x) : \quad \text{subject to } f_i(x) \geq 0 \quad i = 1, \dots, k$$

Note that

$$\max_{\alpha > 0} -\alpha f_i(x) = \begin{cases} \infty & f_i(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Equivalent Form:

$$\gamma^* = \min_{x \in \mathbb{F}} \max_{\alpha_i > 0} f_0(x) - \sum_i \alpha_i f_i(x) = \min_{x \in \mathbb{F}} \max_{\alpha_i > 0} L(x, \alpha)$$

The function $L(x, \alpha) = f_0(x) - \sum_i \alpha_i f_i(x)$ is called the **Lagrangian**.

The **Dual Problem** switches the min-max:

$$\lambda^* = \max_{\alpha_i > 0} \min_{x \in \mathbb{F}} f_0(x) - \sum_i \alpha_i f_i(x)$$

Or if we define $g(\alpha) = \min_{x \in \mathbb{F}} f_0(x) - \sum_i \alpha_i f_i(x)$,

$$\lambda^* = \max_{\alpha_i > 0} g(\alpha)$$

For convex optimization, $\lambda^* = \gamma^*$. However, $x^* \neq \alpha^*$.

Note: We always have $\max_x \min_y g(x, y) \leq \min_y \max_x g(x, y)$ (2-player game).

$$\min_{x \in \mathcal{X}} f(x) \quad \text{subject to } f_i(x) \geq 0 \quad i = 1, \dots, k$$

Note that

$$\max_{x \in \mathcal{X}} -\alpha f_i(x) = \begin{cases} \infty & f_i(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Equivalent Form:

$$\gamma^* = \min_{x \in \mathcal{X}} \max_{\alpha_i \geq 0} f(x) - \sum_i \alpha_i f_i(x) = \min_{x \in \mathcal{X}} \max_{\alpha_i \geq 0} L(x, \alpha)$$

The function $L(x, \alpha) = f(x) - \sum_i \alpha_i f_i(x)$ is called the **Lagrangian**.The **Dual Problem** switches the min-max:

$$\lambda^* = \max_{\alpha_i \geq 0} \min_{x \in \mathcal{X}} f(x) - \sum_i \alpha_i f_i(x)$$

Or if we define $g(\alpha) = \min_{x \in \mathcal{X}} f(x) - \sum_i \alpha_i f_i(x)$,

$$\lambda^* = \max_{\alpha_i \geq 0} g(\alpha)$$

For convex optimization, $\lambda^* = \gamma^*$. However, $\lambda^* \neq \alpha^*$.**Note:** We always have $\max_x \min_y g(x, y) \leq \min_{\alpha} \max_x g(x, y)$ (2-player game).

The property (Weak Duality)

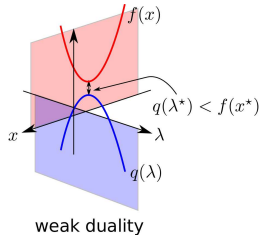
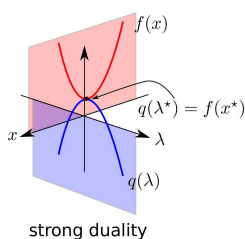
$$\max_x \min_y g(x, y) \leq \min_y \max_x g(x, y)$$

always holds for smooth functions and when equality holds can be interpreted as a Nash equilibrium (See window drawings in “A Beautiful Mind”).

The player which moves second always wins. Note the second player here is the inner optimization problem (min on LHS and max on RHS). To verify, just use the function

$$g(x, y) = xy$$

Strong vs. Weak Duality (In the Lagrangian Sense)



Primal Problem:

$$\gamma^* = \min_{x \in \mathbb{R}} f(x) : \quad x \in S$$

Dual Problem:

$$\lambda^* = \max_{y \in \mathbb{R}} f_D(y) : \quad y \in S_D$$

Definition 4.

Strong Duality holds if $\lambda^* = \gamma^*$. **Weak Duality** holds if $\lambda^* < \gamma^*$.

- Strong Duality holds if f , S are convex and S has non-empty interior.
- Weak Duality always holds.
- The Lagrangian Dual Problem is **ALWAYS** Convex.

Lagrangian Duality Examples

Two Ways to Solve the Same Problem

Primal LP:

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & c^T x : \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Dual LP:

$$\begin{aligned} \min_{y \in \mathbb{R}} \quad & b^T y : \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

$$g(\alpha) = \max_{x \geq 0} c^T x - \alpha^T (Ax - b) = \max_{x \geq 0} (c - A^T \alpha)^T x + \alpha^T b = \begin{cases} b^T \alpha & A^T \alpha \geq c \\ \infty & \text{otherwise} \end{cases}$$

Primal SDP:

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & \sum_{i=1}^m c_i x_i \\ & X = \sum_{i=1}^m F_i x_i - F_0 \geq 0 \end{aligned}$$

Dual SDP:

$$\begin{aligned} \max_{y \in \mathbb{R}} \quad & \text{trace}(F_0 Y) : \\ & \text{trace}(F_i Y) = c_i \quad (i = 1, \dots, m) \\ & Y \geq 0 \end{aligned}$$

The trace notation simply means $\text{trace}(FY) = \sum_{i,j} F_{ij} Y_{ij}$.

Lagrangian Duality Examples

Primal LP:	Dual LP:
$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^T x; \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$	$\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & b^T y; \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$
$g(\alpha) = \max_{x \geq 0} c^T x - \alpha^T (Ax - b) = \max_{x \geq 0} (c - A^T \alpha)^T x + \alpha^T b = \begin{cases} \alpha^T b & A^T \alpha \geq c \\ \infty & \text{otherwise} \end{cases}$	
Primal SDP:	Dual SDP:
$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \sum_{i=1}^m c_i x_i \\ & X = \sum_{i=1}^m F_i x_i - F_0 \succeq 0 \end{aligned}$	$\begin{aligned} \max_{Y \in \mathbb{S}^n} \quad & \text{trace}(F_0 Y) \\ & \text{trace}(F_i Y) = c_i \quad (i = 1, \dots, m) \\ & Y \succeq 0 \end{aligned}$
The trace notation simply means $\text{trace}(FY) = \sum_{i,j} F_{ij} Y_{ij}$.	

$$\begin{aligned} & \max_{x \geq 0, Ax - b \leq 0} c^T x \\ & = \max_{x \geq 0} \min_{\alpha \geq 0} c^T x - \alpha^T (Ax - b) \\ & \leq \min_{\alpha \geq 0} \max_{x \geq 0} c^T x - \alpha^T (Ax - b) \\ & = \min_{\alpha \geq 0} \max_{x \geq 0} (c^T - \alpha^T A)x + \alpha^T b \\ & = \min_{\alpha \geq 0, c^T - \alpha^T A \leq 0} \alpha^T b \\ & = \min_{\alpha \geq 0, c \leq A^T \alpha} b^T \alpha \end{aligned}$$

Convex Cones: What Does Positivity Even Mean?

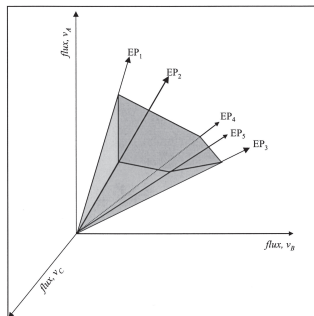
Question: What does $f(x) \geq 0$ mean.

- What does $y \geq 0$ mean?

Definition 5.

A set is a **cone** if for any $x \in Q$,

$$\{\mu x : \mu \geq 0\} \subset Q.$$



Examples:

- **Positive Orthant:** $y \geq 0$ if $y_i \geq 0$ for $i = 1, \dots, n$.
- **Half-space:** $y \geq 0$ if $\sum y_i \geq 0$ ($\mathbf{1}^T y \geq 0$).
 - ▶ More generally, $y \geq 0$ if $\mathbf{a}^T y + b \geq 0$.
- **Intersection of Half-spaces:** $y \geq 0$ if $a_i^T y + b_i \geq 0$ for $i = 1, \dots, n$.
- **Positive Matrices:** $P \geq 0$ if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$.
- **Positive Functions:** $f \geq 0$ if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Generalized Inequalities and Convex Cones

What is an inequality? What does ≥ 0 mean?

- An inequality implies a **partial ordering**:
 - ▶ $x \geq y$ if $x - y \geq 0$
- Any convex cone, C defines a partial ordering:
 - ▶ $x - y \geq 0$ if $x - y \in C$
- The ordering is only partial because $x \not\geq 0$ does not imply $x \leq 0$
 - ▶ $-x \notin C$ does not imply $x \in C$.
 - ▶ x may be indefinite.
- The Cone of Positive Matrices is a partial ordering.
 - ▶ A matrix may have both positive and negative eigenvalues.

Dual Cones (on an inner-product space)

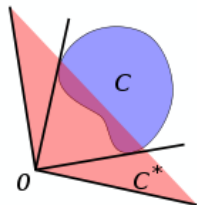
Definition 6.

Two **Sets are Dual** (X and Y) if $x \in X$ implies $\langle x, y \rangle \geq 0$ for all $y \in Y$.

For every point $y \in Y$, the angle between y and every point in X is less than 90° (and vice-versa holds by definition).

We can now consider **Self-Dual Cones**:

- **Positive Orthant:** $y \geq 0$ if $y_i \geq 0$ for $i = 1, \dots, n$.
 - ▶ $\langle x, y \rangle = x^T y = \|x\| \|y\| \cos \theta$.
- **Positive Matrices:** $P \geq 0$ if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$.
 - ▶ $\langle X, Y \rangle = \text{trace}(XY)$



This is why we refer to both primal and dual versions of SDP and LP.

- Their dual problems are of the same form.
- Allows Primal-Dual Algorithms
- Faster Convergence

Linear Algebra Review: Symmetric Matrices

Definition 7.

A square matrix $P \in \mathbb{R}^{n \times n}$ is **Symmetric**, denoted $P \in \mathbb{S}^n$ if $P = P^T$.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 & 3 \\ *^T & 4 & 5 \\ *^T & *^T & 6 \end{bmatrix}$$

- Symmetric Matrices have **Real** Eigenvalues: $\{\lambda : Px = \lambda x, x \in \mathbb{R}^n\} \subset \mathbb{R}$

Definition 8.

A matrix U is **Unitary** (orthogonal) if $U^{-1} = U^T$.

Unitary Matrices have the pleasant property that $\|Ux\| = \|x\|$ for any $x \in \mathbb{R}^n$.

- e.g. Rotation Matrices $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$.

Symmetric Matrices can be diagonalized by a Unitary matrix.

$$P = U\Lambda U^T \quad \text{or} \quad U^T P U = \Lambda \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \quad (\lambda_i \text{ are eigenvalues})$$

Linear Algebra Review: Singular Value Decomposition

For ANY non-symmetric matrices, $P = U\Sigma V^T$, with U, V unitary and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ diagonal, positive.

- This is called the **Singular Value Decomposition (SVD)**.
- $\sigma_i \geq 0$ (**Singular Values**) are the square roots of the eigenvalues of $P^T P$.
- The maximum σ_i is denoted $\bar{\sigma}(P)$. Note that $\bar{\sigma}(P) = \max_x \frac{\|Px\|}{\|x\|}$.

Matlab Code: `> [U,S,V]=svd([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -.33 & -.74 & -.59 \\ -.59 & -.33 & .74 \\ -.74 & .59 & -.33 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 11.34 & 0 & 0 \\ 0 & .52 & 0 \\ 0 & 0 & .17 \end{bmatrix}}_\Sigma \underbrace{\begin{bmatrix} -.33 & .74 & -.59 \\ -.59 & .33 & .74 \\ -.74 & -.59 & -.33 \end{bmatrix}^T}_{V^T}$$

NOTE: This is not quite the same as Diagonalization! Unless....

Matlab Code: `> [U,S]=schur([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -.33 & -.74 & -.59 \\ -.59 & -.33 & .74 \\ -.74 & .59 & -.33 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 11.34 & 0 & 0 \\ 0 & -.52 & 0 \\ 0 & 0 & .17 \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} -.33 & -.74 & -.59 \\ -.59 & -.33 & .74 \\ -.74 & .59 & -.33 \end{bmatrix}^T}_{U^T}$$

Linear Algebra Review: Matrix Positivity - Definition

Try not to define positivity using eigenvalues. (Eigenvalues don't add)

Definition 9.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

Definition 10.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \quad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

Pleasant Properties of Positive Matrices

Lemma 11.

$P \in \mathbb{S}^n$ is positive definite if and only if all its eigenvalues are positive.

In this case, the SVD and Unitary (Schur) Diagonalization are the same.

$$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -.37 & .82 & -.44 \\ -.58 & -.58 & -.58 \\ -.73 & .04 & .69 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 9.4 & 0 & 0 \\ 0 & 3.4 & 0 \\ 0 & 0 & 2.2 \end{bmatrix}}_{\Lambda=\Sigma} \underbrace{\begin{bmatrix} -.37 & .82 & -.44 \\ -.58 & -.58 & -.58 \\ -.73 & .04 & .69 \end{bmatrix}^T}_{U^T=V^T}$$

Fact: If T is invertible, then $P > 0$ is equivalent to $T^T P T > 0$.

- $P > 0 \rightarrow (Tx)^T P (Tx) = x^T T^T P T x > 0$
- $T^T P T > 0 \rightarrow (T^{-1}x)^T T^T P T (T^{-1}x) = x^T P x > 0$

Fact: A Positive Definite matrix is invertible: $P^{-1} = U \Sigma^{-1} U^T$.

Fact: The inverse of a positive definite matrix is positive definite: $\Sigma^{-1} > 0$

Fact: For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ where $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

$$P^{\frac{1}{2}} = U \Sigma^{\frac{1}{2}} U^T > 0 \quad P^{\frac{1}{2}} P^{\frac{1}{2}} = U \Sigma^{\frac{1}{2}} U^T U \Sigma^{\frac{1}{2}} U^T = U \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} U^T = U \Sigma U^T = P$$

Building Linear Matrix Inequalities

Fact: $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, implies both $X > 0$ and $Z > 0$.

Proof: True since $\begin{bmatrix} 0 \\ z \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} > 0$ and $\begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$

Fact: $X > 0$ and $Z > 0$ is equivalent to $\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} > 0$.

Proof: True since $x^T X x > 0$ and $z^T Z z > 0$ implies

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = x^T X x + z^T Z z > 0.$$

Theorem 12 (Schur Complement).

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X - Y Z^{-1} Y^T & 0 \\ 0 & Z \end{bmatrix} > 0$$

Diagonal Dominance: If X and Z are big enough, Y doesn't matter.

Leftover Factoids on Positive Matrices

Things which are true:

- $P > 0$ and $Q > 0$ implies $P + Q > 0$.
- $P > 0$ implies $\mu P > 0$ for any positive scalar $\mu > 0$.
- $M^T M \geq 0$ for any matrix, M .
- $P > 0$ implies $M^T P M > 0$ if nullspace of M is empty.

Things which are **NOT TRUE** (Fallacies):

- $P > 0$ implies $TPT^{-1} > 0$.
- $P > 0$ and $Q > 0$ implies $PQ > 0$.
- $P > 0$ implies $T^T P + PT > 0$
- $P \geq 0$ implies P invertible.
- A has positive eigenvalues implies $A + A^T > 0$. ([1 -3; 0 1])

Semidefinite Programming - Dual Form

$$\begin{array}{ll} \text{minimize} & \text{trace } CX \\ \text{subject to} & \text{trace } A_i X = b_i \quad \text{for all } i \\ & X \succeq 0 \end{array}$$

- The variable X is a symmetric matrix
- $X \succeq 0$ is another way to say X is positive semidefinite
- The feasible set is the intersection of an affine set with the *positive semidefinite cone*

$$\{ X \in \mathbb{S}^n \mid X \succeq 0 \}$$

$$\text{Recall } \text{trace } CX = \sum_{i,j} C_{i,j} X_{j,i}.$$

SDPs with Explicit Variables - Primal Form

We can also explicitly parametrize the affine set to give

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where F_0, F_1, \dots, F_n are symmetric matrices.

The inequality constraint is called a *Linear Matrix Inequality (LMI)*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Linear Matrix Inequalities

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.

Common Form:

Find X :

$$\sum_i A_i X B_i + Q > 0$$

There are several very efficient **LMI/SDP Solvers** which interface with YALMIP:

- **SeDuMi**
 - ▶ Fast, but somewhat unreliable.
 - ▶ Link: <http://sedumi.ie.lehigh.edu/>
- **LMI Lab** (Part of Matlab's Robust Control Toolbox)
 - ▶ Universally disliked, but you already have it.
 - ▶ Link: <http://www.mathworks.com/help/robust/lmis.html>
- **MOSEK** (commercial, but free academic licenses available)
 - ▶ Probably the most reliable
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