LMI Methods in Optimal and Robust Control

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Lecture 4: LMIs for State-Space Internal Stability

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Previous lectures focused on optimization and optimization algorithms per se.

Now, we consider the problem of **Optimization of Systems**. Specifically, properties of the system. There are many ways to view this problem.

- Systems can be defined by differential equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

- We can view the system as a set of constraints on variables $x(t)$, $y(t)$.
- We can view the system as a map from $x(0) \to x(\cdot)$ or $u(\cdot) \to y(\cdot)$.
- Sometimes, we will optimize properties of the matrices A, B, C, D (e.g. Eigenvalues)
  - Sometimes, matrix properties tell us about system performance.
  - Properties such as stability and step-response

We start the lecture by examining the relationship between eigenvalues and system properties.

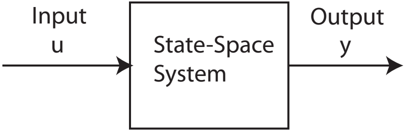
- We then show how LMIs can be used to constrain the eigenvalues of a system.

Solving the Equations

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

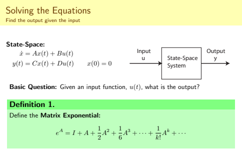


Basic Question: Given an input function, $u(t)$, what is the output?

Definition 1.

Define the **Matrix Exponential:**

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$



The series expansion of the matrix exponential is not an approximation

- This is its definition
- It can be shown that the series converges for any matrix, A.

Properties of the Matrix Exponential

The matrix exponential is similar to the scalar exponential with important differences.

- $e^0 = I$

$$e^0 = I + 0 + \frac{1}{2}0^2 + \frac{1}{6}0^3 + \cdots = I$$

- $e^{M^T} = \left(e^M\right)^T$

$$e^{M^T} = I + M^T + \frac{1}{2}(M^T)^2 + \frac{1}{6}(M^T)^3 + \cdots + \frac{1}{k!}(M^T)^k + \cdots$$

- $\frac{d}{dt}e^{At} = Ae^{At}$

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{k!}(At)^k + \cdots\right) \\ &= A\left(I + (At) + \frac{1}{2}(At)^2 + \cdots + \frac{1}{(k-1)!}(At)^{k-1} + \cdots\right)\end{aligned}$$

- However, $e^{M+N} \neq e^M e^N$ unless, $MN = NM$.

State-Space Solutions and Jordan Blocks

The equation
$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$
has solution
$$x(t) = e^{At}x_0$$

Proof.

Let $x(t) = e^{At}x_0$, then

- $\dot{x}(t) = Ae^{At}x_0 = Ax(t)$.
- $x(0) = e^0 x_0 = x_0$ $\square$

Question: So what does e^{At} look like? Let λ_i be eigenvalues of A

Definition 2.

A **Jordan Block**, J_i is a matrix which looks like
$$J_i = \lambda_i I + N = \underbrace{\begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix}}_{\lambda_i I} + \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}}_{N}$$

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- Therefore, the system
$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$
defines a map from $x_0 \to x(\cdot)$.
- Stability properties of the system are defined using this map.
- How to relate properties of the matrix, A, to properties of the map $x_0 \to x(\cdot)$???
- Lets look at the eigenvalues of A.

The *Jordan Blocks* are the almost diagonal matrices

The matrix $N \in \mathbb{R}^k$ is *Nilpotent*. This means $N^d = 0$ for $d \geq k$.

Thing to Know: the Jordan Decomposition

Theorem 3.

For any $A \in \mathbb{R}^{n\times n}$, there exists an invertible T such that

$$A = TJT^{-1}, J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{bmatrix} \qquad \textit{where } J_i \textit{ are Jordan Blocks}$$

- $A^k = TJ^kT^{-1}$. Hence

$$e^{At} = Te^{Jt}T^{-1} = T \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_n t} \end{bmatrix} T^{-1} = T \begin{bmatrix} e^{(\lambda_1 I + N)t} & & \\ & \ddots & \\ & & e^{(\lambda_n I + N)t} \end{bmatrix} T^{-1}$$

- $\lambda_i I$ and N commute, hence $e^{\lambda_i I + N} = e^{\lambda_i I} e^N$. Further $N^d = 0$ for $d \geq k$.

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & & \\ & \ddots & \\ & & e^{\lambda_i t} \end{bmatrix} \left[1 + Nt + \frac{1}{2}N^2 t^2 + \cdots + \frac{1}{(k-1)!} N^{k-1} t^{k-1} \right]$$

- $\lim_{t\to\infty} t^i e^{\lambda t} = 0$ if and only if $\operatorname{Re}\lambda < 0$

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- $\lim_{t\to\infty} t^i e^{\lambda t} = 0$ if and only if $\operatorname{Re}\lambda < 0$

Not all matrices are diagonalizable.

- All matrices have a Jordan decomposition.

We have shown that every element of e^{At} has the form $t^d e^{\lambda_i(A)t}$, where $\lambda_i(A)$ are the eigenvalues of A

- Shows that the system map $x_0 \mapsto e^{At}x_0$ is stable if and only if all the eigenvalues of A have negative real part.

Stability of Continuous and Discrete-Time Systems

Definition 4.

A is **Hurwitz** if $\mathrm{Re}\,\lambda_i(A) < 0$ for all i.

Theorem 5.

$\dot{x}(t) = Ax(t)$ is stable if and only if A is Hurwitz.

For **Discrete-Time Systems:** $x_{k+1} = Ax_k$,

$$x_k = A^k x_0$$

Definition 6.

A is **Schur** if $|\lambda_i(A)| < 1$ for all i.

Theorem 7.

$x_{k+1} = Ax_k$ is stable if and only if A is Schur.

Stability of Continuous and Discrete-Time Systems

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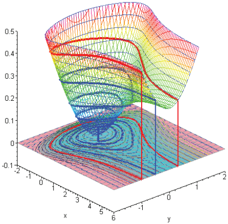
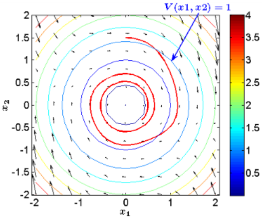
Theorem 7.
$x_{k+1} = Ax_k$ is stable if and only if A is Schur.

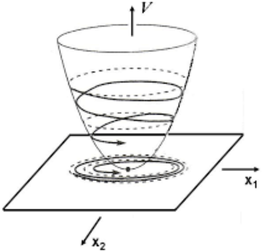
We have shown that the system property of *stability* can be characterized by the eigenvalues of A.

- But eigenvalues are difficult to compute, much less optimize.
- How to use LMIs to characterize the eigenvalues of A???

Thing to Know: Lyapunov Functions Prove Global Stability







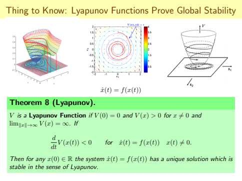
$$\dot{x}(t) = f(x(t))$$

Theorem 8 (Lyapunov).

*V is a **Lyapunov Function** if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $\lim_{\|x\| \to \infty} V(x) = \infty$. If*

$$\frac{d}{dt}V(x(t)) < 0 \qquad \text{for} \quad \dot{x}(t) = f(x(t)) \quad x(t) \neq 0.$$

Then for any $x(0) \in \mathbb{R}$ the system $\dot{x}(t) = f(x(t))$ has a unique solution which is stable in the sense of Lyapunov.



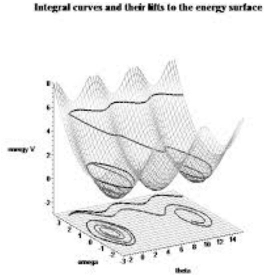
Lyapunov functions generalize the notion of energy of a system.

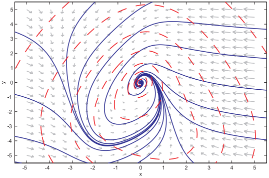
- Sometimes, we think of kinetic and potential energy
  - This is misleading. Better to think of Lyapunov functions as the map from state x_0 to some property of the solution $x(t) = e^{At}x_0$.
  - Typically, The property we are interested in is the square integral of $x(t)$.

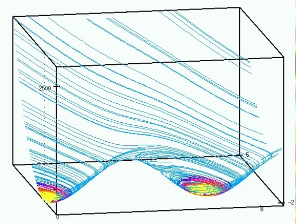
$$V(x_0) = \int_0^\infty \|x(t)\|^2 dt = \int_0^\infty \|e^{At}x_0\|^2 dt$$

- More on this later.

Lyapunov Functions: They Also Prove LOCAL Stability







$$\dot{x}(t) = f(x(t))$$

Theorem 9 (Lyapunov Stability).

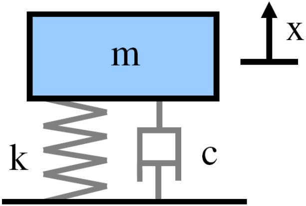
Suppose there exists a continuous V *and* $\alpha, \beta, \gamma > 0$ *where*

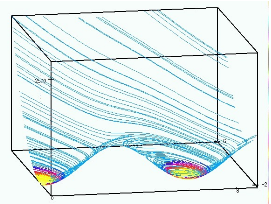
$$\beta\|x\|^2 \le V(x) \le \alpha\|x\|^2$$

$$\dot{V}(x(t)) = \nabla V(x(t))^T f(x(t)) \le -\gamma\|x(t)\|^2$$

for all $x \in X$. *Then any sub-level set of* V *in* X *is a* **Domain of Attraction**.

Examples of Lyapunov Functions





Mass-Spring:

$$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$$

$$V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\begin{aligned}\dot{V}(x) &= \dot{x}(-c\dot{x} - kx) + kx\dot{x} \\ &= -c\dot{x}^2 - k\dot{x}x + kx\dot{x} \\ &= -c\dot{x}^2 \le 0\end{aligned}$$

Pendulum:

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 \quad \dot{x}_1 = x_2$$

$$V(x) = (1 - \cos x_1)gl + \frac{1}{2}l^2x_2^2$$

$$\begin{aligned}\dot{V}(x) &= glx_2 \sin x_1 - glx_2 \sin x_1 \\ &= 0\end{aligned}$$

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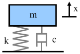

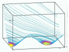

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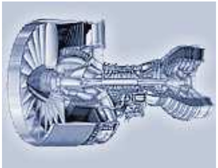
Note that in both these cases, the Lyapunov function is the combined "potential" and "kinetic" energy

- As mentioned, this is not a good way to think about Lyapunov functions.
- Only works for mechanical problems solely under the influence of a conservative field.
- This is rarely the case.
- Provides the wrong intuition.

An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).

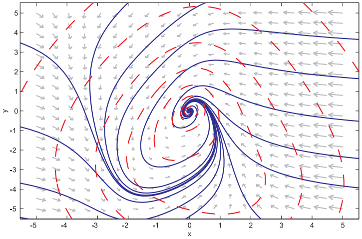


$$\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3$$

$$\dot{y} = 3x - y$$

This is feasible with

$$V(x) = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2$$
$$+ 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.090723y^4$$



The Lyapunov Inequality (Our First LMI)

Lemma 10 (An LMI for Hurwitz Stability).

A is Hurwitz if and only if there exists a $P > 0$ such that

$$A^T P + PA < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P + PA < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T A^T P x(t) + x(t)^T P A x(t) \\ &= x(t)^T \left(A^T P + PA \right) x(t) \end{aligned}$$

- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

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- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

- Although all problems can be posed as optimization problems, not all optimization problems are solvable. In particular, the problem must be convex; the constraints typically should be in the form of a convex cone or inequality. Unfortunately, the constraint that the eigenvalues of a matrix lie in the left half-plane is not easy to enforce directly. The eigenvalues of a matrix are a complex function of the elements of the matrix and, in addition, are quite sensitive to errors in those elements. For this reason, we seek to reformulate the constraint that a matrix be Hurwitz.
- Unlike eigenvalues, the set of Lyapunov functions is a convex cone. This means it is by definition an inequality and therefore well-suited to numerical optimization. Furthermore, the restriction to quadratic Lyapunov functions is likewise a cone constraint and finally, there is a 1-1 relationship between positive matrices and positive Lyapunov functions. This is because the definition of positivity we defined for matrices is identical to the definition of positivity we defined for Lyapunov functions.
- Our first LMI provides the kernel by which we will transmute problems which involve the placement of eigenvalues into problems on the feasibility of certain LMIs. We will use variations of this proof throughout the course, with perhaps the culmination being the KYP Lemma.

The Lyapunov Inequality

Proof.

For the other direction, if A is Hurwitz, for any $Q > 0$, let

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds$$

- Converges because A is Hurwitz.
- Furthermore
$$\begin{aligned}
PA &= \int_0^\infty e^{A^T s} Q e^{As} A ds \\
&= \int_0^\infty e^{A^T s} Q A e^{As} ds = \int_0^\infty e^{A^T s} Q \frac{d}{ds}\left(e^{As}\right) ds \\
&= \left[e^{A^T s} Q e^{As}\right]_0^\infty - \int_0^\infty \frac{d}{ds} e^{A^T s} Q e^{As} \\
&= -Q - \int_0^\infty A^T e^{A^T s} Q e^{As} = -Q - A^T P
\end{aligned}$$
- Thus $PA + A^T P = -Q < 0$.

The Lyapunov Inequality

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- Converges because A is Hurwitz.
- Furthermore
$$\begin{aligned} PA &= \int_0^\infty e^{A^T s} Q e^{As} A ds \\ &= \int_0^\infty e^{A^T s} Q A e^{As} ds = \int_0^\infty e^{A^T s} Q \frac{d}{ds}\left(e^{As}\right) ds \\ &= \left[e^{A^T s} Q e^{As}\right]_0^\infty - \int_0^\infty \frac{d}{ds} e^{A^T s} Q e^{As} \\ &= -Q - \int_0^\infty A^T e^{A^T s} Q e^{As} = -Q - A^T P \end{aligned}$$
- Thus $PA + A^T P = -Q < 0$.

$\square$

- Obviously, $e^{As}A = Ae^{As}$
- Second line uses integration by parts.
- Third line uses $\lim_{s\to\infty} e^{As} = 0$

- Sufficiency is easy. Obviously P defines a Lyapunov function in the obvious way. Now simply expand the $\dot{V}$ and we are done.
- Necessity is not as obvious. It comes from the fact that one interpretation of a Lyapunov function is that it represents the magnitude of the forward-time solution starting from a point x. Since $x(t) = e^{At}x_0$, we then have

$$V(x_0) = \int_0^\infty x(s)^T x(s) ds = \int_0^\infty x_0^T e^{A^T s} e^{As} x_0 ds = x_0^T P x_0$$

Discrete-Time Lyapunov Functions

$$x_{k+1} = f(x_k)$$

Theorem 11 (Lyapunov).

V is a **Lyapunov Function** *if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ and $\lim_{\|x\| \to \infty} V(x) = \infty$. If*

$$V(x_{k+1}) < V(x_k) \qquad \textit{whenever} \quad x_{k+1} = f(x_k) \quad x_k \neq 0,$$

then for any $x_0 \in \mathbb{R}^n$ the system $x_{k+1} = f(x_k)$ is stable in the sense of Lyapunov.

Discrete-Time Lyapunov Functions

Lemma 12 (An LMI for Schur Stability).

A is Schur if and only if there exists a $P > 0$ such that

$$A^T P A - P < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P A - P < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} V(x_{k+1}) &= x_{k+1}^T P x_{k+1} \\ &= x_k^T A^T P A x_k \\ &< x_k^T P x_k = V(x_k) \end{aligned}$$

- Hence $V(x_{k+1}) < V(x_k)$ for all $k \geq 0$. Thus the system is Stable.
- Stability implies A is Schur.

Lyapunov Functions

Proof.

For the other direction, if A is Hurwitz, for any $Q > 0$, let

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k$$

$$\begin{aligned} A^T P A - P &= \sum_{k=1}^{\infty} (A^T)^k Q A^k - \sum_{k=0}^{\infty} (A^T)^k Q A^k \\ &= -(A^T)^0 Q A^0 = -Q < 0 \end{aligned}$$

- Thus $A^T P A - P < 0$. $\square$

YALMIP Code:

```
> P = sdpvar(n); eta=.1;
> F=[P>=eta*eye(n)];
> F=[F; A' P A - P<=0];
> optimize(F);
```

Proof.

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> optimize(F);
```

- Necessity in the discrete-time case is similar to the continuous-time case. Now, however, the solution is a sequence and its size is defined in the ℓ_2 sense. $x_k = A^k x_0$, we then have

$$V(x_0) = \|x\|_{\ell_2} = \sum_{i=0}^{\infty} x_0^T (A^T)^i A^i x_0$$

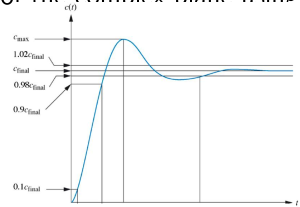
- Note the nonlinearity in A. This will be a problem which will require fixing. For now, however, A is fixed, so the matrix inequality is linear.

Pole Locations AKA D-stability

Some people still care about pole locations.

- For these people, we have D-stability.

To begin, you have to define the acceptable region of the complex plane using inequality constraints.



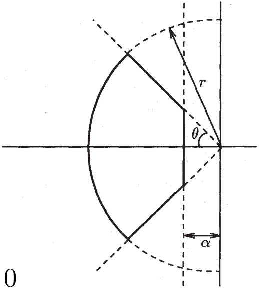
- **Rise Time:** $\omega_n \le \frac{1.8}{t_r}$
- **Settling Time:** $\sigma \le -\frac{4.6}{t_s}$
- **Percent Overshoot:** $\sigma \le -\frac{\ln M_p}{\pi}|\omega_d|$

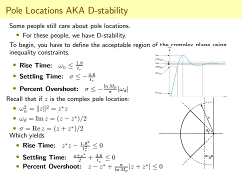
Recall that if z is the complex pole location:

- $\omega_n^2 = \|z\|^2 = z^* z$
- $\omega_d = \operatorname{Im} z = (z - z^*)/2$
- $\sigma = \operatorname{Re} z = (z + z^*)/2$

Which yields

- **Rise Time:** $z^* z - \frac{1.8^2}{t_r^2} \le 0$
- **Settling Time:** $\frac{z+z^*}{2} + \frac{4.6}{t_s} \le 0$
- **Percent Overshoot:** $z - z^* + \frac{\pi}{\ln M_p}|z + z^*| \le 0$





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- Now its time to start introducing lots of LMIs and ways to define them. Up to now, we have been very theory-oriented. This was the minimal amount of theory needed to allow you to understand the next few pages. The next few pages are more practical. Still hard, however.
- The goal, again, is to translate constraints on the eigenvalues of A (which are horribly nonlinear and non-convex functions of the elements of A) to the feasibility of matrix inequalities which are linear in A.
- Note that none of these approximations are valid unless the system is composed of only 2 poles and no zeros.
- See, e.g. Franklin, Powell, Enami for derivations.

An LMI for Pole Locations

Gutman proposed a nice LMI for D-stability with a single constraint

Theorem 13 (Gutman).

The eigenvalues of A satisfy

$$\lambda_i(A) \in \left\{ z \in \mathbb{C} \,:\, \sum_{k,l} c_{kl} z^k (z^*)^l < 0 \right\}$$

if and only if there exists some $P > 0$ such that

$$\sum_{k,l} c_{kl} A^k P (A^T)^l < 0$$

But this has some disadvantages

- There can only be one constraint.
- The LMI is not linear in A (But IS linear in P).
  - So controller synthesis is not an LMI.

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- Simply replace z with A and put a P in the middle!
- Region need not be convex!
- ANY polynomial constraint on z can be represented in this way.

An LMI for Convex Regions of the Complex Plane

To get around the limitations of Gutman's result, we introduce the concept of LMI regions.

- These are regions which can be represented using LMIs in the z and z^* variables

Definition 14.

An **LMI Region** of the complex plane has the form

$$\{z \in \mathbb{C} \;:\; F_0 + zF_1 + z^*F_2 < 0\}$$

Such regions are hard to visualize, but

- Are convex
  - e.g. Minimum rise time is not allowed!
- Can intersect multiple convex regions

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- Are convex
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- Can intersect multiple convex regions

- Remember multiple LMI constraints can be concatenated on the diagonal to obtain a single larger LMI constraint.
- Recall the feasible set of an LMI is called the spectahedron!

Never Forget.... the Schur Complement!

Theorem 15 (Schur Complement).

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X - YZ^{-1}Y^T & 0 \\ 0 & Z \end{bmatrix} > 0$$

Rise Time: $z^*z - \underbrace{\frac{1.8^2}{t_r^2}}_{r^2} \le 0$

$$\begin{bmatrix}-r & z\\ z^* & -r\end{bmatrix} = \underbrace{\begin{bmatrix}-r & 0\\ 0 & -r\end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}}_{F_2} z^* < 0$$

Which by the Schur complement is equivalent to $r - z^*r^{-1}z > 0$.

Settling Time: $\frac{4.6}{t_s} + \frac{z+z^*}{2} \le 0$

Percent Overshoot: $|z - z^*| + \frac{\pi}{\ln M_p}(z + z^*) \le 0$

$$\begin{bmatrix}\pi(z+z^*) & \ln M_p(z-z^*)\\ \ln M_p(z-z^*)^* & \pi(z+z^*)\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix}\pi & \ln M_p\\ -\ln M_p & \pi\end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix}\pi & -\ln M_p\\ \ln M_p & \pi\end{bmatrix}}_{F_2} z^*$$

Which by the Schur complement is equivalent to $z + z^* < 0$ and $(z - z^*)^2 - (\frac{\pi}{\ln M_p})^2|z + z^*|^2 > 0$.

An LMI for Convex Regions of the Complex Plane

Examples

Rise Time: $z^*z - \underbrace{\frac{1.8^2}{t_r^2}}_{r^2} \leq 0$

$$\begin{bmatrix} -r & z \\ z^* & -r \end{bmatrix} = \underbrace{\begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{F_2} z^* < 0$$

Which by the Schur complement is equivalent to $r - z^*r^{-1}z > 0$.

Settling Time: $\frac{4.6}{t_s} + \frac{z+z^*}{2} \leq 0$

Percent Overshoot: $|z - z^*| + \frac{\pi}{\ln M_p}(z + z^*) \leq 0$

$$\begin{bmatrix} \pi(z+z^*) & \ln M_p(z-z^*) \\ \ln M_p(z-z^*)^* & \pi(z+z^*) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} \pi & \ln M_p \\ -\ln M_p & \pi \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} \pi & -\ln M_p \\ \ln M_p & \pi \end{bmatrix}}_{F_2} z^*$$

Which by the Schur complement is equivalent to $z + z^* < 0$ and $(z-z^*)^2 - (\frac{\pi}{\ln M_p})^2|z+z^*|^2 > 0$.

- For rise time $r - z^*r^{-1}z > 0$ is equivalent to $r^2 - z^*z > 0$
- For PO, recall $a > b$ is equivalent to $a^2 > b^2$ if $a, b > 0$ as the square is monotonic increasing

$$\begin{bmatrix} -\pi(z+z^*) & -\ln M_p(z-z^*) \\ -\ln M_p(z-z^*)^* & -\pi(z+z^*) \end{bmatrix} \geq 0 \qquad (-\pi(z+z^*) > 0)$$

$$-\pi(z+z^*) - (\ln M_p)^2(z-z^*)^*(z-z^*)\frac{1}{-\pi(z+z^*)} \geq 0$$

$$\pi^2(z+z^*)^2 - (\ln M_p)^2(z-z^*)^*(z-z^*) \geq 0 \qquad (\text{since } z+z^* < 0)$$

$$-\pi^2(z+z^*)^2 + (\ln M_p)^2(z-z^*)^*(z-z^*) \leq 0$$

$$-\left(\frac{\pi}{\ln M_p}\right)^2 (z+z^*)^2 + |z-z^*|^2 \leq 0$$

$$|z-z^*|^2 \leq \left(\frac{\pi}{\ln M_p}\right)^2 (|z+z^*|)^2$$

$$\left(\frac{\pi}{\ln M_p}\right)|z+z^*| \geq |z-z^*|$$

An LMI for Convex Regions of the Complex Plane

Theorem 16 (Chilali + Gahinet).

The pole locations, $z \in \mathbb{C}$ of A satisfy

$$z \in \{z \in \mathbb{C} \;:\; F_0 + zF_1 + z^* F_2 < 0\}$$

if and only if there exists some $P > 0$ such that

$$F_0 \otimes P + F_1 \otimes (AP) + F_2 \otimes (AP)^T < 0$$

The notation $F \otimes P$ is Kronecker notation and means for each element of Fz, replace the scalar z with the matrix P. So, e.g.

$$\underbrace{\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}}_{F_0} \otimes P := \begin{bmatrix} f_{11}P & f_{12}P \\ f_{12}P & f_{22}P \end{bmatrix}$$

An LMI for Sector Regions of the Complex Plane

Rise Time:

$$\begin{bmatrix} -r & z \\ z^* & -r \end{bmatrix} = \underbrace{\begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}}_{F_0} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{F_1} z + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{F_2} z^* < 0$$

becomes

Lemma 17.

The pole locations, $z \in \mathbb{C}$ of A satisfy $z^* z \le r^2$ if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} -rP & AP \\ (AP)^T & -rP \end{bmatrix} < 0$$

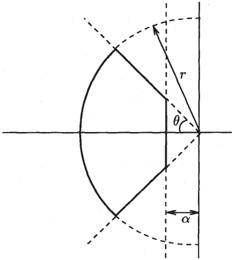
Settling Time: $\frac{4.6}{t_s} + z + z^* \le 0$
becomes

Lemma 18.

The pole locations, $z \in \mathbb{C}$ of A satisfy $2\,\mathrm{Re}\, x \le -\alpha$ if and only if there exists some $P > 0$ such that

$$AP + (AP)^T + \alpha P < 0$$

An LMI for Sector Regions of the Complex Plane



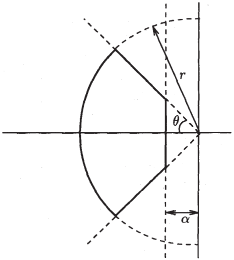
Percent Overshoot: $z + z^* \leq -\frac{\ln M_p}{\pi}|z - z^*|$

Lemma 19.

The pole locations, $z \in \mathbb{C}$ of A satisfy $z + z^* \leq -\frac{\ln M_p}{\pi}|z - z^*|$ if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} \pi(AP + (AP)^T) & \ln M_p(AP - (AP)^T) \\ \ln M_p(AP - (AP)^T)^T & \pi(AP + (AP)^T) \end{bmatrix} < 0$$

A Combined LMI for D-stability



Theorem 20.

The pole locations, $z \in \mathbb{C}$ *of* A *satisfy* $z^* z \le r^2$, $\operatorname{Re} x \le -\alpha$ *and* $z + z^* \le -\frac{\ln M_p}{\pi}|z - z^*|$ *if and only if there exists some* $P > 0$ *such that*

$$\begin{bmatrix} -rP & AP \\ (AP)^T & -rP \end{bmatrix} < 0,$$

$$AP + (AP)^T + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} \pi(AP + (AP)^T) & \ln M_p(AP - (AP)^T) \\ \ln M_p(AP - (AP)^T)^T & \pi(AP + (AP)^T) \end{bmatrix} < 0$$

Discrete-Time Lyapunov Functions

Next Time: LMIs for Controllability and Observability

Discussion Question: Can we do D-stability for discrete-time systems???

- Unfortunately, it is not so easy to translate properties of the transient to eigenvalue locations for discrete time systems.
- Possible Course Project or Wikibooks addition?