LMIs in Systems Analysis and Control

Matthew M. Peet
Arizona State University

Lecture 06: LMIs for Observability and the Luenberger Observer

Observability

Consider a system with no input:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$
$$y(t) = Cx(t)$$

Definition 1.

For a given T, the pair (A, C) is **Observable** on $[0, T]$ if, given $y(t)$ for $t \in [0, T]$, we can reconstruct x_0.

Definition 2.

Given (C, A), the flow map, $\Psi_T : \mathbb{R}^p \to \mathbb{F}(\mathbb{R}, \mathbb{R}^p)$ is

$$\Psi_T : x_0 \mapsto Ce^{At}x_0 \qquad t \in [0, T]$$

So $y = \Psi_T x_0$ means $y(t) = Ce^{At}x_0$.

Proposition 1.

The pair (C, A) is observable if and only if Ψ_T is invertible, which implies

$$\ker \Psi_T = 0$$

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Again, the system defines a map from $x_0 \mapsto y(t) = Ce^{At}x_0$.

- The system (C, A) is invertible if the only state which gets mapped to zero is zero.

Observability

Definition 3.

The **Observability Matrix**, $O(C,A)$ is defined as

$$O(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem 4.

$$\ker \Psi_T$$
$$= \ker C \cap \ker CA \cap \ker CA^2 \cap \cdots \cap \ker CA^{n-1}$$
$$= \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Definition 5.

The **Unobservable Subspace** is $N_{CA} = \ker \Psi_T = \ker O(C,A)$.

Theorem 6.

For a given pair (C,A), the following are equivalent

- $\ker Y = 0$
- $\ker \Psi_T = 0$
- $\ker O(C,A) = 0$

$Y = \int_0^\infty e^{A^T s} C^T C e^{As} ds$ *is the Observability Grammian.*

If the state is observable, then it is observable arbitrarily fast.

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If the state is observable, then it is observable arbitrarily fast.

Again: we relate

- Properties of the map $x_0 \to y(\cdot)$
- Properties of a matrix, $O(C,A)$
- Existence of a Positive matrix, Y

The Observability Gramian

Definition 7.

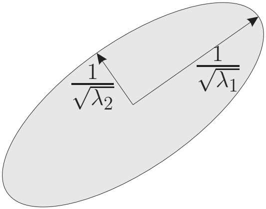
For pair (C, A), the **Observability Grammian** is defined as

$$Y = \Psi_T^* \Psi_T = \int_0^\infty e^{A^T s} C^T C e^{As} ds$$

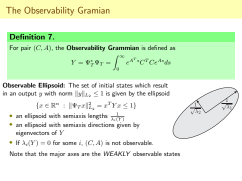
Observable Ellipsoid: The set of initial states which result in an output y with norm $\|y\|_{L_2} \leq 1$ is given by the ellipsoid

$$\{x \in \mathbb{R}^n \ : \ \|\Psi_T x\|_{L_2}^2 = x^T Y x \leq 1\}$$

- an ellipsoid with semiaxis lengths $\frac{1}{\lambda_i(Y)}$
- an ellipsoid with semiaxis directions given by eigenvectors of Y
- If $\lambda_i(Y) = 0$ for some i, (C, A) is not observable.



Note that the major axes are the *WEAKLY* observable states



Note the duality.

- For the controllability and observability grammians. largest singular values are strongly controllable in case a) (length λ) and strongly observable in case b) (length $1/\lambda$).
- Theorem 6 describes conditions under which the kernel is 0, while the controllability equivalent describes conditions under which image is full rank (kernel of transpose is 0)
- In fact, controllability and observability are duals. This is formalized in the following slide.

$$x_0^T Y x_0 = \int_0^\infty x_0^T e^{A^T s} C^T C e^{As} x_0 ds = \|C e^{At} x_0\|_{L_2}^2$$

Duality

The Controllability and Observability matrices are related

$$O(C,A) = C(A^T, C^T)^T$$
$$C(A,B) = O(B^T, A^T)^T$$

For this reason, the study of controllability and observability are related.

$$N_{CA} = \ker O(C,A) = [\operatorname{image} C(A^T, C^T)]^\perp = C^\perp_{A^T C^T}$$
$$C_{AB} = \operatorname{image} C(A,B) = [\ker O(B^T, A^T)]^\perp = N^\perp_{A^T B^T}$$

We can investigate observability of (C,A) by studying controllability of (A^T, C^T)

- (C,A) is observable if and only if (A^T, C^T) is controllable.

Lemma 8 (An LMI for the Observability Gramian).

(C,A) is observable iff $Y > 0$ is the unique solution to

$$A^T Y + YA + C^T C = 0$$

Recall $W > 0$ and $AW + WA^T + BB^T = 0$ for controllability!

Duality

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Note the duality.

- For a given subspace X, the orthogonal complement is denoted

$$X^\perp := \{y \ : \ y^T x = 0 \text{ for all } \quad x \in X\}$$

- From linear algebra, $\operatorname{image} M = (\ker M^T)^\perp$

Recall $u \mapsto x(\cdot)$

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

And $x_0 \mapsto y(\cdot)$

$$y(t) = Ce^{At} x_0$$

The duality relationship follows by using the expansion

$$e^A = e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k$$

where by Cayley-Hamilton, we can stop at $k = N$

Observers (aka Estimators)

Suppose we have designed a controller

$$u(t) = Kx(t)$$

but we can only measure $y(t) = Cx(t)$!

Question: How to find $x(t)$?

- If (C, A) observable, then we can observe $y(t)$ on $t \in [t, t+T]$.
  - But by then its too late!
  - we need $x(t)$ in *real time*!

Definition 9.

An **Observer**, is an *Artificial Dynamical System* whose output tracks $x(t)$.

Suppose we want to observe the following system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Lets assume the observer is state-space

- What are our inputs and output?
- What is the dimension of the system?

Observers

Inputs: $u(t)$ and $y(t)$.
Outputs: Estimate of the state: $\hat{x}(t)$.

Assume the observer has the same dimension as the system

$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)$$
$$\hat{x}(t) = Qz(t) + Ry(t) + Su(t)$$

We want $\lim_{t\to 0} e(t) = \lim_{t\to 0} x(t) - \hat{x}(t) = 0$

- for any u, $z(0)$, and $x(0)$.
- We would also like internal stability, etc.

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- We would also like internal stability, etc.

- Actually, if the dimension of the observer is any less (even $n-1$) than that of the original system, the problem becomes NP-hard.
- This is the most general case of an observer. In this lecture, we will reduce the number of variables. However, we will later return to the most general form in order to improve performance.

Coupled System and Error Dynamics

System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Observer Dynamics:

$$\dot{z}(t) = Mz(t) + Ny(t) + Pu(t)$$
$$\hat{x}(t) = Qz(t) + Ry(t) + Su(t)$$

DYNAMICS Of The Error: What are the dynamics of $e(t) = x(t) - \hat{x}(t)$?

$$\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - Q\dot{z}(t) + R\dot{y}(t) + S\dot{u}(t) \\
&= Ax(t) + Bu(t) - Q(Mz(t) + Ny(t) + Pu(t)) + R(C\dot{x}(t) + D\dot{u}(t)) + S\dot{u}(t) \\
&= Ax(t) + Bu(t) - QMz(t) - QN(Cx(t) + Du(t)) - QPu(t) \\
&\qquad\qquad + RC(Ax(t) + Bu(t)) + (S + RD)\dot{u}(t) \\
&= (A + RCA - QNC)e(t) + (AQ + RCAQ - QNCQ - QM)z(t) \\
&\quad + (A + RCA - QNC)Ry(t) + (B + RCB - QP - QND)u(t) + (S + RD)\dot{u}(t)
\end{aligned}$$

Designing an observer requires that these dynamics are Hurwitz.

The Luenberger Observer

For now, we consider a special kind of observers, parameterized by the matrix L

$$\begin{aligned}\dot{z}(t) &= (A + LC)z(t) - Ly(t) + (B + LD)u(t) \\ &= \underbrace{Az(t) + Bu(t)}_{\text{Propagate estimate}} + \underbrace{L(\underbrace{Cz(t) + Du(t)}_{\text{Predicted Output}} - \underbrace{y(t)}_{\text{Actual Output}})}_{\text{Correction Term}}\end{aligned}$$

$$\hat{x}(t) = z(t)$$

In the general formulation, this corresponds to

$$\begin{aligned} &M = A + LC; \qquad N = -L; \qquad P = B + LD; \\ &Q = I; \qquad R = 0; \qquad S = 0; \end{aligned}$$

So in this case $z(t) = \hat{x}(t)$ and the error dynamics simplify to

$$\dot{e}(t) = (A + LC)e(t)$$

Thus the criterion for convergence is $A + LC$ Hurwitz.

Question Can we choose L such that $A + LC$ is Hurwitz?
Similar to choosing $A + BF$.

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Similar to choosing $A+BF$.

- The estimator state is itself the estimate of the state.
- Recalling

$$\begin{gathered} M = A + LC; \qquad N = -L; \qquad P = B + LD; \\ Q = I; \qquad R = 0; \qquad S = 0; \end{gathered}$$

- $(A + RCA - QNC) = QM = A + LC$
- $(A + RCA - QNC)R = 0$
- $AQ + RCAQ - QNCQ - QM = 0$
- $B + RCB - QP - QND = 0$
- $S + RD = 0$

Observability

If turns out that observability and detectability are useful

Theorem 10.

The eigenvalues of $A + LC$ are freely assignable through L if and only if (C, A) is observable.

If we only need $A + LC$ Hurwitz, then the test is easier.

- We only need detectability

Theorem 11.

An observer exists if and only if (C, A) is detectable

Note: Theorem applies to ANY observer, not just Luenberger observers.

- This implies Luenberger observers are necessary and sufficient for design of stable estimators.

An LMI for Observer Synthesis

Question: How to compute L?

- The eigenvalues of $A+LC$ and $(A+LC)^T = A^T + C^T L^T$ are the same.
- This is the same problem as controller design!

Theorem 12.

There exists a K such that $A+BK$ is stable if and only if there exists some $P>0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0,$$

where $K = ZP^{-1}$.

Theorem 13.

There exists an L such that $A+LC$ is stable if and only if there exists some $P>0$ and Z such that

$$A^T P + PA + C^T Z + Z^T C < 0,$$

where $L = P^{-1} Z^T$.

So now we know how to design an Luenberger observer.

- Also called an estimator

The error dynamics will be dictated by the eigenvalues of $A+LC$.

- generally a good idea for the observer to converge faster than the plant.

Observer-Based Controllers

Summary: What do we know?

- How to design a controller which uses the full state.
- How to design an observer which converges to the full state.

Question: Is the combined system stable?

- We know the error dynamics converge.
- Lets look at the coupled dynamics.

Proposition 2.

The system defined by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
u(t) &= K\hat{x}(t) \\
\dot{\hat{x}}(t) &= (A + LC)\hat{x}(t) - Ly(t) + (B + LD)u(t) \\
&= A\hat{x}(t) + Bu(t) + L(C\hat{x}(t) + Du(t) - y(t)) \\
&= (A + LC + BK + LDK)\,\hat{x}(t) - Ly(t)
\end{aligned}$$

has eigenvalues equal to that of $A + LC$ and $A + BK$.

Observer-Based Controllers

The proof is relatively easy

Proof.

The state dynamics are

$$\dot{x}(t) = Ax(t) + BK\hat{x}(t)$$

Rewrite the estimation dynamics as

$$\begin{aligned}
\dot{\hat{x}}(t) &= (A + LC + BK + LDK)\,\hat{x}(t) - Ly(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,K\hat{x}(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + (B + LD)\,u(t) - LCx(t) - LDu(t) \\
&= (A + LC)\,\hat{x}(t) + Bu(t) - LCx(t) \\
&= (A + LC + BK)\,\hat{x}(t) - LCx(t)
\end{aligned}$$

In state-space form, we get

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + LC + BK \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

Proof.

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix} = \begin{bmatrix}A & BK\\ -LC & A+LC+BK\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix}$$

Use the similarity transform $T = T^{-1} = \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}$.

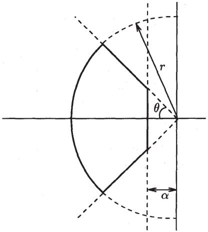
$$\begin{aligned}T\bar{A}T^{-1} &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A & BK\\ -LC & A+LC+BK\end{bmatrix}\begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\\ &= \begin{bmatrix}I & 0\\ I & -I\end{bmatrix}\begin{bmatrix}A+BK & -BK\\ A+BK & -(A+LC+BK)\end{bmatrix}\\ &= \begin{bmatrix}A+BK & -BK\\ 0 & A+LC\end{bmatrix}\end{aligned}$$

which has eigenvalues $A+LC$ and $A+BK$. □

Eigenvalues are invariant under similarity transforms.

An LMI for Observer D-Stability

- Use the Controller Synthesis LMI to choose K.
- Then use the following LMI to choose L.
- If both $A + LC$ and $A + BK$ satisfy the D-stability condition, then the eigenvalues of the close-loop system will as well.



Lemma 14 (An LMI for D-Observer Design).

Suppose there exists $X > 0$ *and* Z *such that*

$$\begin{bmatrix} -rP & (PA+ZC)^T \\ PA+ZC & -rP \end{bmatrix} < 0,$$

$$(PA+ZC)^T + PA + ZC + 2\alpha P < 0, \quad \textit{and}$$

$$\begin{bmatrix} ((PA+ZC)^T + PA + ZC) & c((PA+ZC)^T - (PA+ZC)) \\ c(PA+ZC-(PA+ZC)^T) & ((PA+ZC)^T + PA + ZC) \end{bmatrix} < 0$$

Then if $L = P^{-1}Z$*, the pole locations,* $z \in \mathbb{C}$ *of* $A + LC$ *satisfy* $|x| \le r$*,* $\operatorname{Re} x \le -\alpha$ *and* $z + z^* \le -c|z - z^*|$*.*

One Step Discrete-Time Observers

Nominal System:

$$x_{k+1} = Ax_k + Bu_k \qquad y_k = Cx_k + Du_k$$

Observer Format:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(C\hat{x}_k + Du_k - y_k)$$

This gives error ($e_k = x_k - \hat{x}_k$) dynamics

$$e_{k+1} = (A + LC)e_k$$

So the Problem is exactly the same as for the continuous-time case.

Separation Principle for Discrete-Time Observers

Closed-Loop System Dynamics:

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BK\hat{x}_k$$

Closed-Loop Estimator Dynamics:

$$\begin{aligned}\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L(C\hat{x}_k + Du_k - y_k)\\ &= A\hat{x}_k + BK\hat{x}_k + L(C\hat{x}_k + DK\hat{x}_k - Cx_k - DK\hat{x}_k)\\ &= (A + BK + LC)\hat{x}_k - LCx_k\end{aligned}$$

Coupled System-Estimator Dynamics

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$$

Which (as for continuous-time) has eigenvalues $A + LC$ and $A + BK$.

Summary of LMIs Learned

Table 6.1 Criteria for Stabilizability of (A, B)

	Criteria	Variables
Hurwitz	$P > 0$ $AP + PA^T + BW + W^TB^T < 0$	P, W
	$P > 0$ $N_b^T \left(AP + PA^T\right) N_b < 0$	P
	$P > 0$ $AP + PA^T - BB^T < 0$	P
Schur	$\begin{bmatrix} -P & AP + BW \\ PA^T + W^TB^T & -P \end{bmatrix} < 0$	P, W
	$\begin{bmatrix} -N_b^T P N_b & N_b^T AP \\ PA^T N_b & -P \end{bmatrix} < 0$	P
	$\begin{bmatrix} -P & PA^T \\ AP & -P - BB^T \end{bmatrix} < 0$	P

Table 6.2 Criteria for Detectability of (A, B, C)

	Criteria	Variables
Hurwitz	$P > 0$ $A^TP + PA + W^TC + C^TW < 0$	P, W
	$P > 0$ $N_c^T \left(A^TP + PA\right) N_c < 0$	P
	$P > 0$ $A^TP + PA - C^TC < 0$	P
Schur	$\begin{bmatrix} -P & A^TP + C^TW^T \\ PA + WC & -P \end{bmatrix} < 0$	P, W
	$\begin{bmatrix} -N_c^T P N_c & N_c^T A^T P \\ PAN_c & -P \end{bmatrix} < 0$	P
	$\begin{bmatrix} -P & PA \\ A^TP & -P - C^TC \end{bmatrix} < 0$	P

Examples:

Example 6.2: Jet Aircraft $\dot{x} = Ax + Bu$ and $y = Cx$.

$$A = \begin{bmatrix} -.0558 & -.9968 & .0802 & .0415 \\ .5980 & -.1150 & -.0318 & 0 \\ -3.0500 & .388 & -.465 & 0 \\ 0 & .0805 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} .0729 & .0001 \\ -4.75 & 1.23 \\ 1.53 & 10.63 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 6.3: Discrete-Time System $x_{k+1} = Ax_k + Bu_k$ and $y = Cx_k$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$