LMI Methods in Optimal and Robust Control

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Lecture 07: LMIs for the H_2 and H_∞ Norms of Systems and Transfer Functions

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- We now begin to cover subjects not covered in 506. This rather long section of the course covers lectures 7-11. Lecture 10 is the culmination and describes the H_∞-optimal output feedback problem. Lecture 11 is brief and extends this to the H_2-optimal output feedback problem.
- This lecture covers the mathematical machinery which will then be used in Lectures 8-11.
- The essence of the lecture is to take what you learned about transfer functions and block diagrams as an undergrad and make them mathematically rigorous.

Definition 1.

A **Norm** on a vector space, V, is a function $\|\cdot\| : V \to \mathbb{R}^+$ such that

1. $\|x\| = 0$ if and only if $x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$
3. $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in V$ (Pythagorean Inequality)

Definition 2.

A vector space with an associated norm is called a **Normed Space**.

On infinite sequences $g : \mathbb{N} \to \mathbb{R}$

- $\|g\|_{\ell_1} = \sum_{i=1}^{\infty} |g_i|$
- $\|g\|_{\ell_2} = \sqrt{\sum_{i=1}^{\infty} g_i^2}$
- $\|g\|_{\ell_p} = \sqrt[p]{\sum_{i=1}^{\infty} g_i^p}$
- $\|g\|_{\ell_\infty} = \max_{i=1,\ldots,\infty} |g_i|$

On functions $f : [0, \infty) \to \mathbb{R}$

- $\|f\|_{L_1} = \int_0^{\infty} |f(s)| ds$
- $\|f\|_{L_2} = \sqrt{\int_0^{\infty} f(s)^2 ds}$
- $\|f\|_{L_p} = \sqrt[p]{\int_0^{\infty} f(s)^p ds}$
- $\|f\|_{L_\infty} = \sup_{s \in [0,\infty)} |f(s)|$

Signal Spaces

Normed Spaces and L_p norms

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- In this lecture, we define new spaces of signals and new spaces of systems.
- As an aid, think of signals as a generalization of vectors and systems as a generalization of matrices.
- Think of the Laplace transform as a Unitary coordinate transformation.

There is a distinction between system behaviour and system representation.

- State space A, B, C, D or the transfer function is a representation of a system. These formats uses matrices or complex-valued functions (a signal) to parameterize the representation. However, the system is NOT a set of matrices, nor a transfer function. The system is a set of behaviours. When we talk about system norms, we talk about properties of these behaviours and not properties of A, B, C, D or the transfer function. Yet, when we do optimal control, we must be able to use the representation to infer properties of the system behaviour. H_∞ and H_2, thus are both signal norms and measure the size of the transfer function in a certain sense. However, they are not obviously system norms. The only norm we have for systems is the induced norm, for which systems form an algebra. It is necessary for systems to form an algebra, otherwise the standard use of block-diagram algebra is invalid. It turns out the H_∞ norm is equal to the induced norm of the system. Thus if all systems have finite H_∞ norm, block diagram algebra is valid. However, finite H_2 norm does not necessarily imply finite H_∞ norm. Thus, if we want to interconnect subsystems, we must add finite H_∞ norm as a constraint on each connected component. For optimal H_2 control, we must always add a finite H_∞ norm constraint to the problem. This is true for both LQR and Kalman filtering/LQG problems.

Only L_2 has an **Inner Product**, $\langle \cdot, \cdot \rangle$. Like the dot product, an Inner product defines a measure of angle.

$$\langle x, y\rangle = \|x\|\|y\| \cos\theta$$

So x and y are $\perp$ if $\langle x, y\rangle = 0$

Theorem 3 (Cauchy Schwartz).

If $\|x\|^2 = \langle x, x\rangle$, *then*

$$|\langle x, y\rangle| \le \|x\|\|y\|$$

Likewise $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ iff $\langle x, y\rangle = 0$ (Pythagorean Theorem).

Definition 4.

$L_2[0,\infty)$ is the Hilbert space of functions $f : \mathbb{R}^+ \to \mathbb{R}^n$ with inner product

$$\langle u, y\rangle_{L_2} = \frac{1}{2\pi}\int_0^\infty u(t)^T v(t) dt, \qquad \|u\|_{L_2}^2 = \langle u, u\rangle_{L_2} = \int_0^\infty \|u(t)\|^2 dt$$

Signal Spaces - Inner Products

L_2 is a Hilbert space

Only L_2 has an **Inner Product**, $\langle \cdot, \cdot \rangle$. Like the dot product, an Inner product defines a measure of angle.

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- Don't confuse the Cauchy-Schwartz inequality $\langle x, y \rangle \le \|x\|\|y\|$ defined on inner product spaces (e.g. signals) with the submultiplicative inequality $\|AB\| \le \|A\|\|B\|$ defined for algebras (e.g. Matrices and Systems).

Operator Theory: Linear Operators

A Banach Space is a normed space which is complete (Any L_p or ℓ_p space)
A Hilbert Space is an inner product space which is complete (Only L_2)

Definition 5 (The Space of Systems).

The normed space of bounded linear operators from X to Y is denoted $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ with norm

$$\|P\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Px\|_Y}{\|x\|_X} = K$$

- Satisfies the properties of a norm
- This type of norm is called an "induced" norm
- Notation: $\mathcal{L}(X) := \mathcal{L}(X, X)$
- If X is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space

Properties: Suppose $G_1 \in \mathcal{L}(X, Y)$ and $G_2 \in \mathcal{L}(Y, Z)$

- Then $G_2 \odot G_1 \in \mathcal{L}(X, Z)$.
- $\|G_2 \odot G_1\|_{\mathcal{L}(X,Z)} \leq \|G_2\|_{\mathcal{L}(Y,Z)} \|G_1\|_{\mathcal{L}(X,Y)}$.
- Composition forms an *algebra*.

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- Composition forms an *algebra*.

- An algebra is a Normed Vector space with a multiplication operation $G_2 \odot G_1$ and an identity element (e.g. Matrices and Systems (Linear Operators, more generally)). The algebra depends on which norm you pick.
- Linear Operators act like matrices, except on signals (not vectors)
  - "Linear Algebra" can refer to both operators and matrices
- The induced norm changes with the norms defined on X and Y.
- Recall $\bar{\sigma}(X) = \max_x \frac{\|Px\|_2}{\|x\|_2}$, which is the norm induced by the Euclidean norm on vectors.
- Frobenius norm on Matrices

$$\|P\|_F = \sqrt{\sum_{ij} P_{ij}^2}$$

is not induced by any norm, so does not represent properties of the Matrix as an operator. However, it does allow us to define the inner product on a matrix.

Laplace Transform

Definition 6.

Given $u \in L_2[0, \infty)$, the Laplace Transform of u is $\hat{u} = \Lambda u$, where

$$\hat{u}(s) = (\Lambda u)(s) = \lim_{T \to \infty} \int_0^T u(t) e^{-st} dt$$

if this limit exists.

Λ is a *bounded linear operator* - $\Lambda \in \mathcal{L}(L_2, H_2)$.

- $\Lambda : L_2 \to H_2$.
- The norm $\|\Lambda\|_{\mathcal{L}(L_2, H_2)}$ is

$$\|\Lambda\| = \sup_{u \in L_2} \frac{\|\Lambda u\|_{H_2}}{\|u\|_{L_2}} = ???$$

WAIT! What is H_2?

H_2 - A Space of Integrable Analytic Functions

Definition 7.

A complex function is **analytic** if it is continuous and bounded.

A function is analytic if the Taylor series converges everywhere in the domain.

Definition 8.

A function $\hat{u} : \bar{\mathbb{C}}^+ \to \mathbb{C}^n$ is in H_2 if

1. $\hat{u}(s)$ is analytic on the Open RHP (denoted $\mathbb{C}^+$)
2. For almost every real ω,

$$\lim_{\sigma \to 0^+} \hat{u}(\sigma + \imath\omega) = \hat{u}(\imath\omega)$$

   - Which means continuous up to the imaginary axis

3. $$\int_{-\infty}^{\infty} \sup_{\sigma \geq 0} \|\hat{u}(\sigma + \imath\omega)\|_2^2 < \infty$$

   - Which means integrable on every vertical line.

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   - Which means integrable on every vertical line.

A rational function $\hat{u}(s)$ is analytic in the RHP if it has no poles in the RHP.

Equivalence Between H_2 and L_2

Theorem 9 (Maximum Modulus).

An analytic function cannot obtain its extrema in the interior of the domain.

Hence if $\hat{u}$ satisfies 1) and 2), then

$$\int_{-\infty}^{\infty} \sup_{\sigma \geq 0} \|\hat{u}(\sigma + \imath\omega)\|_2^2 = \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|_2^2 d\omega$$

We equip H_2 with a norm and inner product

$$\|\hat{u}\|_{H_2} = \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|_2^2 d\omega, \qquad \langle \hat{u}, \hat{y} \rangle_{H_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\imath\omega)^* \hat{v}(\imath\omega) d\omega$$

Theorem 10 (Paley-Wiener).

1. *If $u \in L_2[0, \infty)$, then $\Lambda u \in H_2$.*
2. *If $\hat{u} \in H_2$, then there exists a $u \in L_2[0, \infty)$ such that $\hat{u} = \Lambda u$ (**Onto**).*

- Shows that H_2 is exactly the image of Λ on $L_2[0, \infty)$
- Shows the map is invertible

Lemma 11.

$$\langle \Lambda u, \Lambda y \rangle_{H_2} = \langle u, y \rangle_{L_2}$$

- *Thus* Λ *is unitary.*
- $L_2[0, \infty)$ *and* H_2 *are isomorphic.*

Definition 12.

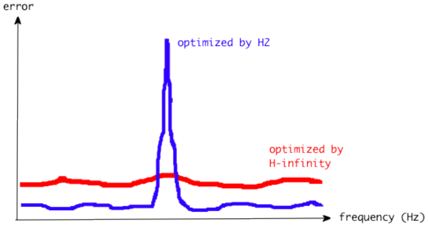
The inverse of the Laplace transform, $\Lambda^{-1} : H_2 \to L_2[0, \infty)$ is

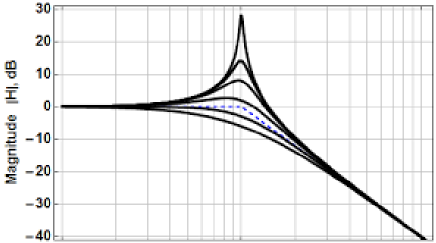
$$u(t) = (\Lambda^{-1}\hat{u})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sigma t} \cdot e^{\imath \omega t}\hat{u}(\sigma + \imath\omega) d\omega$$

where σ can be any real number.

$$\|\Lambda\| = \sup_{u \in L_2} \frac{\|\Lambda u\|_{H_2}}{\|u\|_{L_2}} = ???$$

H_∞ - A Space of Bounded Analytic Functions





Definition 13.

A function $\hat{G} : \bar{\mathbb{C}}^+ \to \mathbb{C}^{n\times m}$ is in H_∞ if

1. $\hat{G}(s)$ is analytic on the CRHP, $\mathbb{C}^+$.
2. $\lim_{\sigma\to 0^+} \hat{G}(\sigma + \imath\omega) = \hat{G}(\imath\omega)$
3. $\sup_{s\in\mathbb{C}^+} \bar{\sigma}(\hat{G}(s)) < \infty$

- H_∞ is a **Banach Space** with norm

$$\|\hat{G}\|_{H_\infty} = \text{ess}\sup_{\omega\in\mathbb{R}} \bar{\sigma}(\hat{G}(\imath\omega))$$

H_∞ (A Signal Space) and Multiplier Operators

Every element of H_∞ defines a multiplication operator.

Definition 14.

Given $\hat{G} \in H_\infty$, define $M_{\hat{G}} \in \mathcal{L}(H_\infty)$ by

$$(M_{\hat{G}}\hat{u})(s) = \hat{G}(s)\hat{u}(s)$$

for $\hat{u} \in H_2$.

Functions vs. Operators

- $\hat{G}$ is a *function*.
- $M_{\hat{G}}$ is an *operator*.

For any analytic functions, $\hat{u}$ and $\hat{G}$, the function

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

is analytic.

- Thus $M_{\hat{G}} : H_2 \rightarrow H_2$.
- Thus $\Lambda^{-1}M_{\hat{G}}\Lambda$ maps $L_2[0,\infty) \rightarrow L_2[0,\infty)$.

Theorem 15.

G is a Causal, Linear, Time-Invariant Operator on L_2 if and only if there exists some $\hat{G} \in H_\infty$ such that $G = \Lambda^{-1}M_{\hat{G}}\Lambda$.

$$(\Lambda G u)(\imath\omega) = \hat{G}(\imath\omega)\hat{u}(\imath\omega)$$

H_∞ is the space of **transfer functions** for linear time-invariant systems.

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H_∞ is the space of **transfer functions** for linear time-invariant systems.

We have said *Nothing* about the structure or parameterization of G.

- This result is **NOT** limited to state-space systems
- Also applies to PDEs, Delay systems, fractional systems, et c.

H_∞ - The space of "Transfer Functions"

From Paley-Wiener, if $G = \Lambda^{-1} M_{\hat{G}} \Lambda$

Theorem 16.

$$\|G\|_{\mathcal{L}(L_2)} = \|M_{\hat{G}}\|_{\mathcal{L}(H_2)} = \|\hat{G}\|_{H_\infty}$$

The **Gain** of the system G can be calculated as $\|\hat{G}\|_{H_\infty}$

- This is the motivation for H_∞ control
- minimize $\sup_u \frac{\|Gu\|_{L_2}}{\|u\|_{L_2}}$.
    - minimize maximum energy of the output.

Conclusion: H_∞ provides a complete parametrization of the space of causal bounded linear time-invariant operators.

Rational Transfer Functions (RH_∞)

The space of bounded analytic functions, H_∞ is infinite-dimensional.

- this makes it hard to design optimal controllers.

We usually restrict ourselves to state-space systems and state-space controllers.

Definition 17.

The space of rational functions is defined as

$$R := \left\{ \frac{p(s)}{q(s)} \; : \; p, q \text{ are polynomials} \right\}$$

We define the following rational subspaces.

$$RH_2 = R \cap H_2$$
$$RH_\infty = R \cap H_\infty$$

Note that RH_2 and RH_∞ are not **complete**(Banach) spaces.

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All proper, rational transfer functions have a state-space representation.

- The implication is that the limit of a sequence of state-space systems may not be a state-space system.
- For example, the sequence of rational Padé functions converges to $e^{\tau s}$.
- The limit of a set of state-space systems here is a delayed system.

Rational Transfer Functions (RH_∞)

RH_∞ is the set of proper rational functions with no poles in the closed right half-plane (CRHP).

Definition 18.

- A rational function $r(s) = \frac{p(s)}{q(s)}$ is **Proper** if the degree of p is less than or equal to the degree of q.
- A rational function $r(s) = \frac{p(s)}{q(s)}$ is **Strictly Proper** if the degree of p is less than the degree of q.

Proposition 1.

1. $\hat{G} \in RH_\infty$ *if and only if* $\hat{G}$ *is proper with no poles on the closed right half-plane.*

State-Space Systems

Define a **State-Space System** $G : L_2 \to L_2$ by $y = Gu$ if

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t).$$

Theorem 19.

- *For any stable state-space system, G, there exists some $\hat{G} \in RH_\infty$ such that*

$$G = \Lambda^{-1} M_{\hat{G}} \Lambda$$

- *For any $\hat{G} \in RH_\infty$, the operator $G = \Lambda^{-1} M_{\hat{G}} \Lambda$ can be represented in state-space for some A, B, C and D where A is Hurwitz.*

For state-space system, (A, B, C, D),

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

State-Space is NOT Unique. For any invertible T,

- $\hat{G} = C(sI - A)^{-1}B + D = CT^{-1}(sI - TAT^{-1})^{-1}TB + D.$
  - (A, B, C, D) and $(TAT^{-1}, TB, CT^{-1}, D)$ both represent the system G.

The KYP Lemma (AKA: The Bounded Real Lemma)

The most important theorem in this class.

Lemma 20.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \le \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Can be used to calculate the H_∞-norm of a system

- Originally used to solve LMI's using graphs. (Before Computers)
- Now used directly instead of graphical methods like Bode.

The feasibility constraints are linear

- Can be combined with other methods.

The KYP Lemma

Proof.

We will only show that ii) implies i). The other direction requires the Hamiltonian, which we have not discussed.

- We will show that if $y = Gu$, then $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$.
- From the 1 x 1 block of the LMI, we know that $A^T X + XA < 0$, which means A is Hurwitz.
- Because the inequality is strict, there exists some $\epsilon > 0$ such that

$$
\begin{aligned}
&\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \\
&= \begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} < 0
\end{aligned}
$$

- Let $y = Gu$. Then the state-space representation is

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
\dot{x}(t) &= Ax(t) + Bu(t) \qquad\qquad x(0) = 0
\end{aligned}
$$

The KYP Lemma

Proof.

- Let $V(x) = x^T X x$. Then the LMI implies

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left[\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} y^T y$$

$$= x^T (A^T X + XA) x + x^T X B u + u^T B^T X x - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= (Ax + Bu)^T X x + x^T X (Ax + Bu) - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2$$

$$= \dot{V}(x(t)) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2 \leq 0$$

The KYP Lemma

Proof.

- Now we have $\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2 \le 0$
- Integrating in time, we get

$$\int_0^T \left(\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2\right)dt$$
$$= V(x(T)) - V(x(0)) - (\gamma - \epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2\Big) dt \le 0$$

- Because A is Hurwitz, $\lim_{t\to\infty} x(t) = 0$.
- Hence $\lim_{t\to\infty} V(x(t)) = 0$.
- Likewise, because $x(0) = 0$, we have $V(x(0)) = 0$.

□

Proof.

- Since $V(x(0)) = V(x(\infty)) = 0$,

$$\lim_{T\to\infty}\left[\dot V(x(T)) - \dot V(x(0)) - (\gamma-\epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2 dt\right]$$
$$= 0 - 0 - (\gamma-\epsilon)\int_0^\infty \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^\infty \|y(t)\|^2 dt$$
$$= -(\gamma-\epsilon)\|u\|_{L_2}^2 + \frac{1}{\gamma}\|y\|_{L_2}^2 \le 0$$

- Thus

$$\|y\|_{L_2}^2 \le (\gamma^2 - \epsilon\gamma)\|u\|_{L_2}^2$$

- By definition, this means $\|\hat G\|_{H_\infty}^2 = \|G\|_{\mathcal{L}(L_2)}^2 \le (\gamma^2 - \epsilon\gamma) < \gamma^2$ or

$$\|\hat G\|_{H_\infty} < \gamma$$

A Variation on the KYP lemma is the positive-real lemma

Lemma 21.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- G *is passive. i.e.* $(\langle u, Gu \rangle_{L_2} \geq 0)$*.*
- *There exists a* $P > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0$$

The Positive Real Lemma

A Passivity Condition

A Variation on the KYP lemma is the positive-real lemma

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- *There exists a* $P > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0$$

G is Passive (Positive Real) if

$$\hat{G}(s) + \hat{G}(s)^* > 0$$

Or, equivalently $\angle G(s) \in [-90^\circ, +90^\circ]$

- Can be read off the Bode plot
- Phase lead or lag less than 90°

H_2-optimal control

Theorem 22.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. *A is Hurwitz and $\|\hat{P}\|_{H_2}^2 < \gamma$.*
2. *There exists some $X > 0$ such that*

$$\operatorname{trace} CXC^T < \gamma$$
$$AX + XA^T + BB^T < 0$$

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- The H_2 norm of a transfer function is conceptually identical to the Frobenius norm on a matrix.
- Minimizing the H_2-norm using full-state feedback is the LQR problem.
- However, minimizing the H_2 norm reflects a view of the system based on representation and not operation. That is, the controller minimizes the size of the *representation* of the system as opposed to the *performance* of the system.
  - Like judging a book based on how many words it has.
- H_2-Control will require 2 applications of the Schur Complement.
- Note that **a system can have finite H_2 norm and still be unstable!**

Proof.

Suppose A is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Controllability Grammian is defined as

$$X_c = \int_0^\infty e^{At} B B^T e^{A^T} dt$$

Now recall the Laplace transform

$$\begin{aligned}
\left(\Lambda e^{At}\right)(s) &= \int_0^\infty e^{At} e^{-ts} dt \\
&= \int_0^\infty e^{-(sI-A)t} dt \\
&= -(sI-A)^{-1} e^{-(sI-A)t} dt \Big|_{t=0}^{t=-\infty} \\
&= (sI-A)^{-1}
\end{aligned}$$

Hence $\left(\Lambda C e^{At} B\right)(s) = C(sI-A)^{-1}B$. □

Note the implicit assumption on stability.

H_2-optimal control

Proof.

$\left(\Lambda C e^{At} B\right)(s) = C(sI - A)^{-1}B$ implies

$$\begin{aligned}
\|\hat{P}\|_{H_2}^2 &= \|C(sI-A)^{-1}B\|_{H_2}^2 \\
&= \frac{1}{2\pi}\int_0^\infty \text{Trace}((C(\imath\omega I - A)^{-1}B)^*(C(\imath\omega I - A)^{-1}B))d\omega \\
&= \frac{1}{2\pi}\int_0^\infty \text{Trace}((C(\imath\omega I - A)^{-1}B)(C(\imath\omega I - A)^{-1}B)^*)d\omega \\
&= \text{Trace}\int_{-\infty}^\infty Ce^{At}BB^Te^{A^Tt}C^Tdt \\
&= \text{Trace}CX_cC^T
\end{aligned}$$

Thus $X_c \geq 0$ and Trace$CX_cC^T = \|\hat{P}\|_{H_2}^2 < \gamma$. □

Third Equality uses linearity of the trace.
Fourth Equality holds by Plancherel Theorem

Proof.

Likewise Trace$B^T X_o B = \|\hat{P}\|_{H_2}^2$. To show that we can make the inequality strict $X > 0$, we simply let

$$X = \int_0^\infty e^{At} \left(BB^T + \epsilon I \right) e^{A^T} dt$$

for sufficiently small $\epsilon > 0$. Furthermore, we already know the controllability grammian X_c and thus X_ϵ satisfies the Lyapunov inequality.

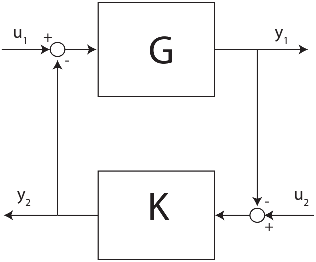
$$A^T X_\epsilon + X_\epsilon A + BB^T < 0$$

These steps can be reversed to obtain necessity. □

System Interconnections

Interconnected Systems (G, K):

$$y_1 = G(u_1 - y_2)$$
$$y_2 = K(u_2 - y_1)$$



Is the interconnection stable?

- If $u_1, u_2 \in L_2$, are $y_1, y_2 \in L_2$?

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -G \\ -K & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} I & G \\ K & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The Matrix Inversion Formula:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Q_1 & -A_{11}^{-1} A_{12} Q_2 \\ -A_{22}^{-1} A_{21} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 & -Q_1 A_{12} A_{22}^{-1} \\ -Q_2 A_{21} A_{11}^{-1} & Q_2 \end{bmatrix}$$

$$Q_1 = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} \qquad Q_2 = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & G \\ K & I \end{bmatrix}^{-1} \begin{bmatrix} G & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (I - GK)^{-1} G & -G(I - KG)^{-1} K \\ -K(I - GK)^{-1} G & (I - KG)^{-1} K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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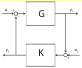

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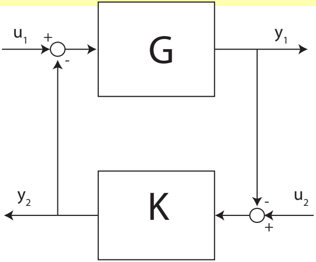
Note how we use the *algebraic nature* of systems to solve for signals.

The Small Gain Theorem

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I-GK)^{-1}G & -G(I-KG)^{-1}K \\ -K(I-GK)^{-1}G & (I-KG)^{-1}K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



If $(I-GK)^{-1}$ is well-behaved, then the interconnection is stable.

- What do we mean by well-behaved?
- $\|(I-GK)^{-1}\| \le \infty$?

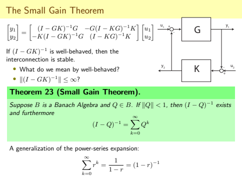
Theorem 23 (Small Gain Theorem).

Suppose B is a Banach Algebra and $Q \in B$. If $\|Q\| < 1$, then $(I-Q)^{-1}$ exists and furthermore

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

A generalization of the power-series expansion:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = (1-r)^{-1}$$

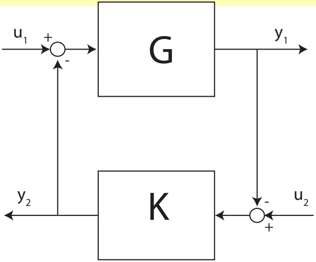


Note the norm here is the one associated to the Banach Algebra

- Systems are only a Banach Algebra under the induced norm, which is equivalent to the H_∞-norm of the transfer function.
- Induced Norms: $\sup_u \frac{\|GKu\|_Y}{\|u\|_X}$
- Therefore: H_∞ norm
- Does the H_2 norm work?????

The Passivity Theorem

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I-GK)^{-1}G & -G(I-KG)^{-1}K \\ -K(I-GK)^{-1}G & (I-KG)^{-1}K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Recall that we say a system G is *passive* if $\langle u, Gu \rangle \geq 0$

- Equivalent to positivity.

Theorem 24 (Passivity Theorem).

If $\langle u, Gu \rangle \geq 0$ (passive), and $-\langle u, Ku \rangle \geq \epsilon \|u\|$ ($-K$ strictly passive), then

$$(I-GK)^{-1}G$$

exists and is passive.

Obvious extensions exist for the other 3 maps:

$$\begin{bmatrix} (I-GK)^{-1}G & -G(I-KG)^{-1}K \\ -K(I-GK)^{-1}G & (I-KG)^{-1}K \end{bmatrix}$$