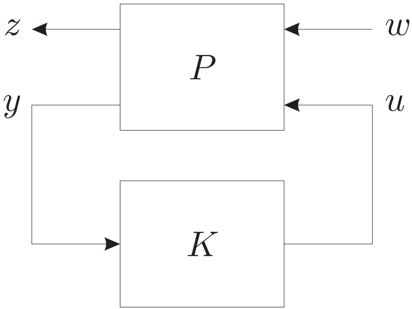
LMI Methods in Optimal and Robust Control

Matthew M. Peet
Arizona State University

Lecture 09: An LMI for H_∞-Optimal Full-State Feedback Control

Recall: Linear Fractional Transformation



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Choose K to minimize

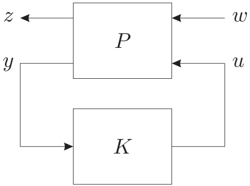
$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right] \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

Optimal Full-State Feedback Control



For the full-state feedback case, we consider a controller of the form

$$u(t) = Fx(t)$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]$$

Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right]$$

Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P}, \hat{K}) = \left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array}\right]$$

By the KYP lemma, $\|\underline{S}(\hat{P}, \hat{K})\|_{H_\infty} < \gamma$ if and only if there exists some $X > 0$ such that

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 \\ B_1^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} (C_1 + D_{12} F)^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} (C_1 + D_{12} F) & D_{11} \end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P},\hat{K}) = \left[\begin{array}{c|c} A+B_2F & B_1 \\ \hline C_1+D_{12}F & D_{11}\end{array}\right]$$

By the KYP lemma, $\|\underline{S}(\hat{P},\hat{K})\|_{H_\infty} < \gamma$ if and only if there exists some $X>0$ such that

$$\begin{bmatrix}(A+B_2F)^TX + X(A+B_2F) & XB_1 \\ B_1^TX & -\gamma I\end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix}(C_1+D_{12}F)^T \\ D_{11}^T\end{bmatrix}\begin{bmatrix}(C_1+D_{12}F) & D_{11}\end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Recall the KYP Lemma

Lemma 1.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \le \gamma$.
- *There exists a* $X>0$ *such that*

$$\begin{bmatrix}A^TX + XA & XB \\ B^TX & -\gamma I\end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix}C^T \\ D^T\end{bmatrix}\begin{bmatrix}C & D\end{bmatrix} < 0$$

Schur Complement

The KYP condition is

$$\begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

Recall the Schur Complement

Theorem 2 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n\times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} < 0$
2. $Q < 0$ *and* $M - RQ^{-1}R^T < 0$

In this case, let $Q = -\frac{1}{\gamma}I < 0$,

$$M = \begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} \qquad R = \begin{bmatrix} C & D \end{bmatrix}^T$$

Note we are making the LMI Larger.

Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the
Full-State Feedback Condition

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & XB_1 & (C_1 + D_{12} F)^T \\ B_1^T X & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F.

Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^TX+XA & XB \\ B^TX & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C^T \\ D^T \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the
Full-State Feedback Condition

$$\begin{bmatrix} (A+B_2F)^TX + X(A+B_2F) & XB_1 & (C_1+D_{12}F)^T \\ B_1^TX & -\gamma I & D_{11}^T \\ (C_1+D_{12}F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in X and F.

Statement of the Dilated KYP Lemma

Lemma 3.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \leq \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^TX+XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

Lemma 4 (KYP Dual).

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $Y > 0$ *such that*

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0$$

Dual KYP Lemma

Proof.

Let $X = Y^{-1}$. Then

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0 \qquad \text{and} \qquad Y > 0$$

if and only if $X > 0$ and

$$\begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

By the Schur complement this is equivalent to

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

By the KYP lemma, this is equivalent to $\|G\|_{H_\infty} \leq \gamma$. □

Full-State Feedback Optimal Control

We can now apply this result to the state-feedback problem.

Theorem 5.

The following are equivalent:

- *There exists an F such that*

$$\left\| \underline{S}\left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right] \right) \right\|_{H_\infty} \le \gamma.$$

- *There exist $Y > 0$ and Z such that*

$$\begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

Then $F = ZY^{-1}$.

Full-State Feedback Optimal Control

Proof.

Suppose there exists an F such that $\|\underline{S}(P, K(0,0,0,F))\|_{H_\infty} \le \gamma$. By the Dual KYP lemma, this implies there exists a $Y > 0$ such that

$$\begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $Z = FY$. Then

$$\begin{bmatrix} YA^T + Z^TB_2^T + AY + B_2Z & B_1 & YC_1^T + Z^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + YF^TB_2^T + AY + B_2FY & B_1 & YC_1^T + YF^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Full-State Feedback Optimal Control

Proof.

Suppose there exists an F such that $\|\underline{\mathsf{S}}(P,K(0,0,0,F))\|_{H_\infty} \le \gamma$. By the Dual KYP lemma, this implies there exists a $Y > 0$ such that

$$\begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $Z = FY$. Then

$$\begin{bmatrix} YA^T + Z^TB_2^T + AY + B_2Z & B_1 & YC_1^T + Z^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix}$$
$$= \begin{bmatrix} YA^T + YF^TB_2^T + AY + B_2FY & B_1 & YC_1^T + YF^TD_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix}$$
$$= \begin{bmatrix} Y(A+B_2F)^T + (A+B_2F)Y & B_1 & Y(C_1+D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1+D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

□

For convenience, we use

$$\underline{\mathsf{S}}(P,K(0,0,0,F)) = \underline{\mathsf{S}}\left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array}\right]\right)$$

Full-State Feedback Optimal Control

Proof.

Now suppose there exists a $Y > 0$ and Z such that

$$\begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let $F = ZY^{-1}$. Then

$$\begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)Y & B_1 & Y(C_1 + D_{12} F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F)Y & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + YF^T B_2^T + AY + B_2 FY & B_1 & YC_1^T + YF^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} FY & D_{11} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Full-State Feedback Optimal Control

Therefore the following optimization problems are equivalent

Form A

$$\min_F \left\| \underline{\mathsf{S}} \left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right] \right) \right\|_{H_\infty}$$

Form B

$$\min_{\gamma, Y, Z} \gamma :$$

$$\begin{bmatrix} -Y & 0 & 0 & 0 \\ 0 & YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ 0 & B_1^T & -\gamma I & D_{11}^T \\ 0 & C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The optimal controller is given by $F = ZY^{-1}$.

Next Lecture: Optimal Output Feedback