LMI Methods in Optimal and Robust Control

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Lecture 11: Relationship between $H_2$, LQG and LGR and LMIs for state and output feedback $H_2$ synthesis
To solve the $H_\infty$-optimal state-feedback problem, we solve

$$\min_{\gamma, X_1, Y_1, A_n, B_n, C_n, D_n} \gamma \quad \text{such that}$$

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X B_1 + B_n D_{21}]^T & -\gamma I \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$
Then, we construct our controller using

\[
D_K = (I + D_{K2}D_{22})^{-1}D_{K2}
\]

\[
B_K = B_{K2}(I - D_{22}D_K)
\]

\[
C_K = (I - D_KD_{22})C_{K2}
\]

\[
A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\]

where

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix} = \begin{bmatrix}
X_2 & X_1B_2 \\
0 & I
\end{bmatrix}^{-1} \begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix} - \begin{bmatrix}
X_1AY_1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
Y_2^T & 0 \\
C_2Y_1 & I
\end{bmatrix}^{-1}.
\]

and where \(X_2\) and \(Y_2\) are any matrices which satisfy \(X_2Y_2^T = I - X_1Y_1\).

- e.g. Let \(Y_2 = I\) and \(X_2 = I - X_1Y_1\).
- The optimal controller is NOT uniquely defined.
- Don’t forget to check invertibility of \(I - D_{22}D_K\)
Conclusion

The $H_\infty$-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$

Minimizes the effect of external input ($w$) on external output ($z$).

$$\|z\|_{L_2} \leq \|S(P, K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain
$H_2$-optimal control

Motivation

$H_2$-optimal control minimizes the $H_2$-norm of the transfer function.

- The $H_2$-norm has no direct interpretation.

\[
\|G\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\omega)^* \hat{G}(\omega)) d\omega
\]

Motivation: Assume external input, $w$, is Gaussian noise with signal variance $S_w$. Then,

\[
E[w(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{S}_w(\omega)) d\omega
\]

**Theorem 1.**

*For an LTI system $P$, if $w$ is noise with spectral density $\hat{S}_w(\omega)$ and $z = Pw$, then $z$ is noise with density*

\[
\hat{S}_z(\omega) = \hat{P}(\omega)\hat{S}(\omega)\hat{P}(\omega)^*
\]
Then the output $z = Pw$ has signal variance (Power)

$$E[z(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\omega)^* S(\omega) \hat{G}(\omega)) d\omega$$

$$\leq \|S\|_{H_\infty} \|G\|_{H_2}^2$$

If the input signal is white noise, then $\hat{S}(\omega) = I$ and

$$E[z(t)^2] = \|G\|_{H_2}^2$$
\(H_2\)-optimal control

Colored Noise

Now suppose the noise is colored with variance \(\hat{S}_w(\omega)\). Now define \(\hat{H}\) as 
\[
\hat{H}(\omega) \hat{H}(\omega)^* = \hat{S}_w(\omega)
\]
and the filtered system.

\[
\hat{P}_s(s) = \begin{bmatrix}
\hat{P}_{11}(s) \hat{H}(s) & \hat{P}_{12}(s) \\
\hat{P}_{21}(s) \hat{H}(s) & \hat{P}_{22}(s)
\end{bmatrix}
\]

Now, applying feedback to the filtered plant, we get

\[
\underline{S}(P_s, K)(s) = P_{11}H + P_{12}(I - KP_{22})^{-1}KP_{21}H = \underline{S}(P, K)H
\]

Now the spectral density, \(\hat{S}_z\) of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

\[
S_z(s) = \underline{S}(P, K)(s)\hat{S}_w(s)\underline{S}(P, K)(s)^*
= \underline{S}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\underline{S}(P, K)(s)^* = \hat{S}(P_s, K)(s)\hat{S}(P_s, K)(s)^*
\]

Thus if \(K\) minimizes the \(H_2\)-norm of the filtered plant \(\|\hat{S}(P_s, K)\|_{H_2}^2\), it will minimize the variance of the true plant under the influence of colored noise with density \(\hat{S}_w\).
Now suppose the noise is colored with variance $\hat{S}_w(\omega)$. Now define $\hat{H}$ as

$$\hat{H}(\omega) = \hat{S}_w(\omega)$$

and the filtered system.

$$\hat{P}(s) = \begin{bmatrix} \hat{P}_{11}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s) & \hat{P}_{22}(s) \end{bmatrix}$$

Now, applying feedback to the filtered plant, we get

$$\hat{S}(P, K)(\omega) = P_{11}H + P_{12}(I - K\hat{P}_{22})^{-1}KP_{21}H = \hat{S}(P, K)H$$

Now the spectral density, $\hat{S}$, of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

$$\hat{S}(\omega) = \hat{S}(P, K)(\omega)\hat{S}(\omega)\hat{S}(P, K)(\omega)^* = \hat{S}(P, K)(\omega)\hat{S}(P, K)(\omega)^*$$

Thus if $K$ minimizes the $H_2$-norm of the filtered plant $[\hat{S}(P, K)]_{H_2}^2$, it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$.

Alternatively, we can write

$$\min_K \|S(P, K)w\|_{var} = \min_K \|S(P, K)Hu\|_{var, u=white} = \min_K \|S(P, K)H\|_{H_2}$$
Theorem 2.

Suppose \( \hat{P}(s) = C(sI - A)^{-1}B \). Then the following are equivalent.

1. \( A \) is Hurwitz and \( \| \hat{P} \|_{H_2} < \gamma \).
2. There exists some \( X > 0 \) such that

\[
\text{trace } CXC^T < \gamma^2
\]
\[
AX + XA^T + BB^T < 0
\]

Recall that the Controllability Grammian is a solution!

- Recall how the proof works.
Proof.

Suppose $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Controllability Grammian is defined as

$$X_c = \int_0^\infty e^{At}BB^T e^{A^T} dt$$

Now recall the Laplace transform

$$(\Lambda e^{At})(s) = \int_0^\infty e^{At} e^{-ts} dt$$

$$= \int_0^\infty e^{-(sI-A)t} dt$$

$$= -(sI-A)^{-1} e^{-(sI-A)t} dt \bigg|_{t=-\infty}^{t=0}$$

$$= (sI-A)^{-1}$$

Hence $(\Lambda Ce^{At}B)(s) = C(sI-A)^{-1}B$. \(\square\)
Proof.

\((\Lambda C e^{At} B)(s) = C(sI - A)^{-1}B\) implies

\[
\|\hat{P}\|_{H_2}^2 = \|C(sI - A)^{-1}B\|_{H_2}^2
\]

\[
= \frac{1}{2\pi} \int_0^\infty \text{Trace}\left((C(\omega I - A)^{-1}B)^*(C(\omega I - A)^{-1}B)\right) d\omega
\]

\[
= \frac{1}{2\pi} \int_0^\infty \text{Trace}\left((C(\omega I - A)^{-1}B)(C(\omega I - A)^{-1}B)^*\right) d\omega
\]

\[
= \text{Trace} \int_{-\infty}^\infty C e^{At} BB^* e^{A^*t} C^* dt
\]

\[
= \text{Trace} CX_c C^T
\]

Thus \(X_c \geq 0\) and \(\text{Trace} CX_c C^T = \|\hat{P}\|_{H_2}^2 < \gamma^2.\)
\section*{H}_2\text{-optimal control}

\textbf{Proof.}

Likewise Trace\(B^T X_0 B = \|\hat{P}\|_{H_2}^2\) where \(X_0\) is the observability Grammian. To show that we can take strict the inequality \(X > 0\), we simply let

\[X = \int_0^\infty e^{At} (BB^T + \epsilon I) e^{A^T} dt\]

for sufficiently small \(\epsilon > 0\). Furthermore, we already know the controllability grammian \(X_c\) and thus \(X_\epsilon\) satisfies the Lyapunov inequality.

\[A^T X_\epsilon + X_\epsilon A + BB^T < 0\]

These steps can be reversed to obtain necessity.
$H_2$-optimal control
Full-State Feedback

Let's consider the full-state feedback problem

$$
\hat{G}(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & 0 & D_{12} \\
I & 0 & 0
\end{bmatrix}
$$

- $D_{12}$ is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

$$
\hat{K}(s) = \begin{bmatrix}
0 & 0 \\
0 & K
\end{bmatrix}
$$
**Theorem 3.**

The following are equivalent.

1. \( \| S(K, P) \|_{H_2} < \gamma \).
2. \( K = ZX^{-1} \) for some \( Z \) and \( X > 0 \) where
   \[
   \begin{bmatrix} A & B_2 \\ Z & Z \end{bmatrix} + \begin{bmatrix} X & Z^T \\ Z & B_2^T \end{bmatrix} + B_1B_1^T < 0
   \]
   \[
   \text{Trace} \left( C_1X + D_{12}Z \right) X^{-1} \left( C_1X + D_{12}Z \right) < \gamma^2
   \]

However, this is nonlinear, so we need to reformulate using the Schur Complement.
Applying the Schur Complement gives the alternative formulation convenient for control.

**Theorem 4.**

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.
2. There exists some $X, W > 0$ such that
   
   $$
   \begin{bmatrix}
   A^T X + XA & XB \\
   B^T X & -\gamma I
   \end{bmatrix} < 0, \quad \begin{bmatrix} X & C^T \end{bmatrix} > 0, \quad \text{Trace} W < \gamma^2
   $$
Theorem 5.

The following are equivalent.

1. $\|S(K, P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ for some $Z$ and $X > 0$ where

$$
\begin{bmatrix}
A & B_2 \\
\end{bmatrix} \begin{bmatrix}
X \\
Z \\
\end{bmatrix} + \begin{bmatrix}
X & Z^T \\
\end{bmatrix} \begin{bmatrix}
A^T \\
B_2^T \\
\end{bmatrix} + B_1B_1^T < 0
$$

$$
\begin{bmatrix}
X \\
C_1X + D_{12}Z \\
\end{bmatrix} \begin{bmatrix}
(C_1X + D_{12}Z)^T \\
W \\
\end{bmatrix} > 0
$$

$\text{Trace} W < \gamma^2$

Thus we can solve the $H_2$-optimal static full-state feedback problem.
The LQR Problem:

- Full-State Feedback
- Choose $K$ to minimize the cost function

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

subject to dynamic constraints

$$\dot{x}(t) = A x(t) + B u(t)$$
$$u(t) = K x(t), \quad x(0) = x_0$$

Trying to minimize the effect of $x_0$ on a weighted-$L_2$-norm of the regulated output.
**H₂-optimal control**

**Relationship to LQR**

To solve the LQR problem using $H₂$ optimal state-feedback control, let

- $C₁ = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$
- $B₂ = B$ and $B₁ = I$.

So that

$$S(\hat{P}, \hat{K}) = \begin{bmatrix} A + B₂K \\ C₁ + D₁₂K \\ B₁ \\ D₁₁ \end{bmatrix} = \begin{bmatrix} A + BK \\ Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \\ 0 \end{bmatrix}$$

And solve the $H₂$ full-state feedback problem. Then if

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$

$$u(t) = Kx(t), \quad x(0) = x₀$$

Then $x(t) = e^{A_{CL}t}x₀$
$H_2$-optimal control

Relationship to LQR

If

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \quad x(0) = x_0$$

then $x(t) = e^{A_{CL}t}x_0$, $u(t) = Ke^{A_{CL}t}x_0$ and

$$\int_0^\infty x(t)^TQx(t) + u(t)^TRu(t)dt = \int_0^\infty x_0^Te^{A_{CL}^Tt}(Q + K^TRK)e^{A_{CL}t}x_0dt$$

$$= \text{Trace} \int_0^\infty x_0^Te^{A_{CL}^Tt} \begin{bmatrix} Q^\frac{1}{2} \\ R^{\frac{1}{2}}K \end{bmatrix}^T \begin{bmatrix} Q^\frac{1}{2} \\ R^{\frac{1}{2}}K \end{bmatrix} e^{A_{CL}t}x_0dt$$

$$= \|x_0\|^2 \text{Trace} \int_0^\infty B_1 e^{A_{CL}^Tt}(C_1 + D_{12}K)^T(C_1 + D_{12}K)e^{A_{CL}t}B_1^Tdt$$

$$= \|x_0\|^2 \|S(K, P)\|_{H_2}^2$$

Thus LQR reduces to a special case of $H_2$ static state-feedback.
Theorem 6 (Scherer, Gahinet).

The following are equivalent.

1. There exists a \( \hat{K} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \) such that \( \|S(K, P)\|_{H^2} < \gamma \).

2. There exist \( X_1, Y_1, Z, A_n, B_n, C_n, D_n \) such that

\[
\begin{bmatrix}
AY_1 + Y_1A^T + B_2C_n + C_n^TB_2^T & \ast^T \\
A^T + A_n + [B_2D_nC_2]^T & X_1A + A^TX_1 + B_nC_2 + C_2^TB_n^T & \ast^T \\
[B_1 + B_2D_nD_{21}]^T & [X_1B_1 + B_nD_{21}]^T & -I \\
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
Y_1 & I & \ast^T \\
I & X_1 & \ast^T \\
C_1Y_1 + D_{12}C_n & C_1 + D_{12}D_nC_2 & Z \\
\end{bmatrix} > 0,
\]

\[D_{11} + D_{12}D_nD_{21} = 0, \quad \text{trace}(Z) < \gamma^2\]
\( H_2\)-optimal output feedback control

As before, the controller can be recovered as

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix} =
\begin{bmatrix} X_2 & X_1B_2 \end{bmatrix}^{-1}
\begin{bmatrix} A_n & B_n \\
C_n & D_n \end{bmatrix} -
\begin{bmatrix} X_1AY_1 & 0 \end{bmatrix}
\begin{bmatrix} Y_2^T & 0 \\
C_2Y_1 & I \end{bmatrix}^{-1}
\]

for any full-rank \( X_2 \) and \( Y_2 \) such that

\[
\begin{bmatrix} X_1 & X_2 \\
X_2^T & X_3 \end{bmatrix} =
\begin{bmatrix} Y_1 & Y_2 \\
Y_2^T & Y_3 \end{bmatrix}^{-1}
\]

To find the actual controller, we use the identities:

\[
D_K = (I + D_{K2}D_{22})^{-1}D_{K2}
\]
\[
B_K = B_{K2}(I - D_{22}D_K)
\]
\[
C_K = (I - D_KD_{22})C_{K2}
\]
\[
A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K
\]
An LMI for Mixed $H_2$-$H_\infty$ optimal output feedback control

**Theorem 7.**

The following are equivalent.

- There exists a $K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ such that $\|S(K, P)\|_{H_2} < \gamma_1$ and $\|S(K, P)\|_{H_\infty} < \gamma_2$.

- There exist $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ such that

\[
\begin{bmatrix}
AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T \\
A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T \\
B_1 + B_2 D_n D_{21} & [X_1 B_1 + B_n D_{21}]^T & -I
\end{bmatrix} < 0,
\]

\[
\begin{bmatrix}
Y_1 & *^T \\
I & \quad \quad X_1 \\
C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z
\end{bmatrix} > 0,
\]

$D_{11} + D_{12} D_n D_{21} = 0$, $\text{trace}(Z) < \gamma_1^2$.
An LMI for the Kalman Filter! - Continuous Time

**System:**
\[
\dot{x} = Ax + Bu + w \\
y = Cx + v
\]

**Filter:**
\[
\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \\
\hat{y} = C\hat{x}
\]

**Error:**
\[
\dot{e}(t) = (A + LC)e(t) + w(t) + Lv(t)
\]

The Kalman Filter chooses \( L \) to minimize the cost \( J = \mathbb{E}[e^T e] \).

\[
L = \Sigma C^T V_2^{-1}
\]

where \( V_1 = \mathbb{E}[w(t)w(t)^T] \) and \( V_2 = \mathbb{E}[v(t)v(t)^T] \) and \( \Sigma \) satisfies

\[
A\Sigma + \Sigma A^T + V_1 = \Sigma C^T V_2^{-1} C\Sigma
\]

If we choose \( u = K\hat{x} \) where \( A + BK \) is stable,

- \( A + LC \) is stable if system is observable (not detectable).
- Closed-Loop is stable by the separation principle (has Luenberger form).
- A Dual to the LQR problem. Replace \((A, B, Q, R, K)\) with \((A^T, C^T, V_1, V_2, L^T)\)
Kalman Filter - Discrete Time

Assume the system is driven by white noise $w_k$ (no feedback)

$$x_{k+1} = Ax_k + Bw_k, \quad y_k =Cx_k$$

The steady-state Kalman filter is an estimator of the form:

$$\hat{x}_{k+1} = A\hat{x}_k + L(C\hat{x}_k - y_k + v_k),$$

where $v_k$ is sensor noise. This gives error ($e_k = x_k - \hat{x}_k$) dynamics

$$e_{k+1} = (A + LC)e_k$$

For the Kalman Filter, we choose $L = A\Sigma C^T (C\Sigma C^T + V)^{-1}$ where $V = E[v_kv_k^T]$ and $\Sigma$ is the steady-state covariance of the error in the estimated state and satisfies

$$\Sigma = A\Sigma A^T + W - A\Sigma C^T (C\Sigma C^T + V)^{-1} C\Sigma A^T$$

where $W = E[w_kw_k^T]$ For the unsteady Kalman filter, $\Sigma_k$ is updated at each time-step.

- If $(A, W)$ controllable and $(C, A)$ observable, then $A + LC$ is stable.
- Again, dual to discrete-time LQR (which we haven’t solved here!)