Lecture 11: Relationship between $H_2$, LQG and LGR and LMIs for state and output feedback $H_2$ synthesis
Conclusion

To solve the $H_\infty$-optimal state-feedback problem, we solve

$$\min_{\gamma, X_1, Y_1, A_n, B_n, C_n, D_n} \gamma \quad \text{such that}$$

$$[X_1 \quad I] > 0$$

$$[AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T \quad *^T \quad *^T \quad *^T \quad *^T \quad *^T \quad *^T] < 0$$

$$[A^T + A_n + [B_2 D_n C_2]^T \quad X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T \quad *^T \quad *^T \quad *^T \quad *^T \quad *^T \quad *^T]$$

$$[B_1 + B_2 D_n D_{21}]^T \quad [X B_1 + B_n D_{21}]^T \quad C_1 Y_1 + D_{12} C_n \quad C_1 + D_{12} D_n C_2 \quad D_{11} + D_{12} D_n D_{21} \quad -\gamma I]$$
Then, we construct our controller using

\[
D_K = (I + D_K D_{22})^{-1} D_{K2} \\
B_K = B_{K2} (I - D_{22} D_K) \\
C_K = (I - D_K D_{22}) C_{K2} \\
A_K = A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K.
\]

where

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix} = \begin{bmatrix}
X_2 & X_1 B_2 \\
0 & I
\end{bmatrix}^{-1} \left[ \begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix} - \begin{bmatrix}
X_1 A Y_1 & 0 \\
0 & 0
\end{bmatrix} \right] \begin{bmatrix}
Y_2^T & 0 \\
C_2 Y_1 & I
\end{bmatrix}^{-1}.
\]

and where \(X_2\) and \(Y_2\) are any matrices which satisfy

\(X_2 Y_2^T = I - X_1 Y_1\).

- e.g. Let \(Y_2 = I\) and \(X_2 = I - X_1 Y_1\).
- The optimal controller is NOT uniquely defined.
- Don’t forget to check invertibility of \(I - D_{22} D_K\)
The $H_\infty$-optimal controller is a dynamic system.

- **Transfer Function** $\hat{K}(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$

Minimizes the effect of external input ($w$) on external output ($z$).

$$\|z\|_{L_2} \leq \|S(P, K)\|_{H_\infty} \|w\|_{L_2}$$

- **Minimum Energy Gain**
$H_2$-optimal control

Motivation

$H_2$-optimal control minimizes the $H_2$-norm of the transfer function.

• The $H_2$-norm has no direct interpretation.

\[ \|G\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(i\omega)^* \hat{G}(i\omega)) d\omega \]

Motivation: Assume external input, $w$, is Gaussian noise with power spectral density $\hat{S}_w$. Then, the variance is given by

\[ E[w(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{S}_w(i\omega)) d\omega \]

**Theorem 1.**

For an LTI system $P$, if $w$ is noise with spectral density $\hat{S}_w(i\omega)$ and $z = Pw$, then $z$ is noise with density

\[ \hat{S}_z(i\omega) = \hat{P}(i\omega)\hat{S}(i\omega)\hat{P}(i\omega)^* \]
Then the output $z = Gw$ has signal variance (Power)

$$E[z(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\omega)^* S(\omega) \hat{G}(\omega)) d\omega$$

$$\leq \|S\|_{H_\infty} \|G\|_{H_2}^2$$

If the input signal is white noise, then $\hat{S}(\omega) = I$ and

$$E[z(t)^2] = \|\hat{G}\|_{H_2}^2$$
Hence the $H_2$ norm represents the power spectral density of the output of the system when the input is white noise.

- Thus $H_2$ optimal control is optimal in a certain sense when the input is expected to be white noise.
- However, this doesn’t work when the noise is colored (concentrated at certain frequencies).
- For colored noise, however, we can use prefilters to obtain optimal controllers.
Now suppose the noise is colored with density $\hat{S}_w(\omega)$. Now define $\hat{H}$ as $\hat{H}(\omega)\hat{H}(\omega)^* = \hat{S}_w(\omega)$ and the filtered system

$$\hat{P}_s(s) = \begin{bmatrix} \hat{P}_{11}(s)\hat{H}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s)\hat{H}(s) & \hat{P}_{22}(s) \end{bmatrix}.$$ 

Now, applying feedback to the filtered plant, we get

$$\mathbb{S}(P_s, K)(s) = P_{11}H + P_{12}(I - KP_{22})^{-1}KP_{21}H = \mathbb{S}(P, K)H$$

Now the spectral density, $\hat{S}_z$ of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

$$\hat{S}_z(s) = \mathbb{S}(P, K)(s)\hat{S}_w(s)\mathbb{S}(P, K)(s)^*$$

$$= \mathbb{S}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\mathbb{S}(P, K)(s)^* = \hat{S}(P_s, K)(s)\hat{S}(P_s, K)(s)^*$$

Thus if $K$ minimizes the $H_2$-norm of the filtered plant ($\|\hat{S}(P_s, K)\|_{H_2}^2$), it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$. 
Now suppose the noise is colored with density \( \hat{S}_w(\omega) \). Now define \( \hat{H} \) as \( \hat{H}(j\omega) = \hat{H}(\omega)^* = \hat{S}_w(\omega) \) and the filtered system

\[
\begin{bmatrix}
\hat{P}_1(s)H(s) & \hat{P}_2(s)
\end{bmatrix}
\]

Now applying feedback to the filtered plant, we get \( S(P,K)(s) = \hat{P}_1H + \hat{P}_2(s)(I - K\hat{P}_2H)^{-1}K\hat{P}_1H = \hat{S}(P,K)H \)

Now the spectral density, \( \hat{S} \), of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

\[
\hat{S}_z(s) = S(P,K)(s) = \hat{S}(P,K)(s)H(s)^* = \hat{S}(P,K)(s)^*H^*(s) = S(P,K)(s)^*H(s)^*
\]

Thus if \( K \) minimizes the \( H^2 \)-norm of the filtered plant \( \|S(P,K)\|_{H_2} \), it will minimize the variance of the true plant under the influence of colored noise with density \( \hat{S}_w \).

In this case, the response of the prefiltered system to white noise is the same as the unfiltered system response to colored noise.

Alternatively, we can write

\[
\min_K \|S(P,K)w\|_{var} = \min_K \|S(P,K)Hu\|_{var} = \min_K \|S(P,K)H\|_{H_2}
\]
Theorem 2.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.
2. There exists some $X > 0$ such that

\[
\text{trace } B^T X B < \gamma^2 \\
A^T X + XA + C^T C < 0
\]

Recall that the Controllability Grammian is a solution!

- Recall how the proof works.
- But this time use the observability grammian.
$H_2$-optimal control

**Proof.**

Suppose $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Observability Grammian is defined as

$$X_o = \int_0^\infty e^{A^T t} C^T C e^{A t} dt$$

Now recall the Laplace transform

$$\left(\Lambda e^{A t}\right)(s) = \int_0^\infty e^{A t} e^{-ts} dt$$

$$= \int_0^\infty e^{-(sI - A)^t} dt$$

$$= -(sI - A)^{-1} e^{- (sI - A)^t} dt \bigg|_{t=-\infty}^{t=0}$$

$$= (sI - A)^{-1}$$

Hence $\left(\Lambda C e^{A t} B\right)(s) = C(sI - A)^{-1} B$.  □
Proof.

\[(\Lambda C e^{At} B) (s) = C(sI - A)^{-1} B \text{ implies} \]

\[
\| \hat{P} \|_{H_2}^2 = \| C(sI - A)^{-1} B \|_{H_2}^2
\]

\[
= \frac{1}{2\pi} \int_{0}^{\infty} \text{Trace} \left( (C(\omega I - A)^{-1} B)^* (C(\omega I - A)^{-1} B) \right) d\omega
\]

\[
= \text{Trace} \int_{-\infty}^{\infty} B^T e^{A^T t} C^T C e^{At} B dt
\]

\[
= \text{Trace} B^T X_o B
\]

Thus \( X_o \geq 0 \) and \( \text{Trace} B^T X_o B = \| \hat{P} \|_{H_2}^2 < \gamma^2 \).

The rest of the proof we can skip.
$H_2$-optimal control

Full-State Feedback

Let's consider the full-state feedback problem

$$\hat{G}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix}$$

- $D_{12}$ is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

$$\hat{K}(s) = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$$
\(H_2\)-optimal control
Full-State Feedback

**Theorem 3.**

The following are equivalent.

1. \(|S(K, P)|_{H_2} < \gamma.\)
2. \(K = ZX^{-1}\) for some \(Z\) and \(X > 0\) where

\[
\begin{bmatrix}
A & B_2 \\
Z & X
\end{bmatrix} + \begin{bmatrix}
X & Z^T \\
B_2^T & A^T
\end{bmatrix} + B_1B_1^T < 0
\]

\[
\text{Trace} \left[ C_1X + D_{12}Z \right] X^{-1} \left[ C_1X + D_{12}Z \right] < \gamma^2
\]

However, this is nonlinear, so we need to reformulate using the Schur Complement.
Applying the Schur Complement gives the alternative formulation convenient for control.

**Theorem 4.**

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. $A$ is Hurwitz and $\|\hat{P}\|_{H^2} < \gamma$.
2. There exists some $X, W > 0$ such that

$$
\begin{bmatrix}
A^T X + XA & XB \\
B^T X & -\gamma I
\end{bmatrix} < 0,
\begin{bmatrix}
X & C^T \\
C & W
\end{bmatrix} > 0,
\text{Trace} W < \gamma^2
$$
Theorem 5.

The following are equivalent.

1. $\|S(K, P)\|_{H^2} < \gamma$.

2. $K = ZX^{-1}$ for some $Z$ and $X > 0$ where

$$
\begin{bmatrix}
  A & B_2 \\
  C_1 & D_{12}
\end{bmatrix}
\begin{bmatrix}
  X \\
  Z
\end{bmatrix}
+ \begin{bmatrix}
  X & Z^T
\end{bmatrix}
\begin{bmatrix}
  A^T \\
  B_2^T
\end{bmatrix}
+ B_1B_1^T < 0
$$

$$
\begin{bmatrix}
  X \\
  C_1X + D_{12}Z
\end{bmatrix}
- \begin{bmatrix}
  (C_1X + D_{12}Z)^T \\
  W
\end{bmatrix} > 0
$$

Trace$W < \gamma^2$

Thus we can solve the $H_2$-optimal static full-state feedback problem.
The LQR Problem:

- Full-State Feedback
- Choose $K$ to minimize the cost function

$$\int_{0}^{\infty} x(t)^T Q x(t) + u(t)^T R u(t) dt$$

subject to dynamic constraints

$$\dot{x}(t) = A x(t) + B u(t)$$
$$u(t) = K x(t), \quad x(0) = x_0$$

Trying to minimize the effect of $x_0$ on a weighted-$L_2$-norm of the regulated output.
$H_2$-optimal control

Relationship to LQR

To solve the LQR problem using $H_2$ optimal state-feedback control, let

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$,
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$ and $D_{11} = 0$,
- $B_2 = B$ and $B_1 = I$.

So that

$$S(P, K) = \begin{bmatrix} A + B_2 K & B_1 \\ C_1 + D_{12} K & D_{11} \end{bmatrix} = \begin{bmatrix} A + BK & I \\ Q^{\frac{1}{2}} & 0 \\ R^{\frac{1}{2}} K \end{bmatrix}$$

And solve the $H_2$ full-state feedback problem. Then if

$$\dot{x}(t) = A_{CL} x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$

$$u(t) = Kx(t), \quad x(0) = x_0$$

Then $x(t) = e^{A_{CL}t}x_0$. 
Translating to the input-output formulation, recall we apply the problem setup to

\[
P = \begin{bmatrix}
A & [B_1 \ B_2] \\
C_1 & D_{11} \ D_{12}
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
0 & 0 \\
0 & K
\end{bmatrix}
\]
Ignoring the regulated outputs for now, we have

\[ \dot{x}(t) = A_{CL} x(t) = (A + BK)x(t) = Ax(t) + Bu(t) \]
\[ u(t) = Kx(t), \quad x(0) = x_0 \]

then \( x(t) = e^{A_{CL}t}x_0, u(t) = Ke^{A_{CL}t}x_0 \) and

\[ \int_0^\infty x(t)^TQx(t) + u(t)^TRu(t)dt = \int_0^\infty x_0^T e^{A_{CL}^Tt}(Q + K^TRK)e^{A_{CL}t}x_0dt \]
\[ = \text{Trace} \int_0^\infty x_0^T e^{A_{CL}^Tt} \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix}^T \begin{bmatrix} Q^{1/2} \\ R^{1/2}K \end{bmatrix} e^{A_{CL}t}x_0dt \]
\[ = \|x_0\|^2 \text{Trace} \int_0^\infty B_1^T e^{A_{CL}^Tt}(C_1 + D_{12}K)^T(C_1 + D_{12}K)e^{A_{CL}t}B_1dt \]
\[ = \|x_0\|^2 B_1^TX_0B_1 = \|x_0\|^2 \|S(P, K)\|_{H_2}^2 \]

Thus LQR reduces to a special case of \( H_2 \) static state-feedback.
H$_2$-optimal control

Recall that

- $C_1 = \begin{bmatrix} Q_1^{\frac{1}{2}} \\ 0 \end{bmatrix}$,

- $D_{12} = \begin{bmatrix} 0 \\ R_1^{\frac{1}{2}} \end{bmatrix}$ and $D_{11} = 0$,

- $B_2 = B$ and $B_1 = I$.

So that

$$S(P, K) = \begin{bmatrix} A + B_2K \\ C_1 + D_{12}K \\ D_{11} \end{bmatrix} = \begin{bmatrix} A + BK \\ Q_1^{\frac{1}{2}} \\ C_1 + D_{12}K \\ R_1^{\frac{1}{2}}K \end{bmatrix}$$

Thus LQR reduces to a special case of $H_2$ static state-feedback.
**Theorem 6 (Scherer, Gahinet).**

The following are equivalent.

- There exists a $\hat{K} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ such that $\|S(K, P)\|_{H_2} < \gamma$.

- There exist $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ such that

$$
\begin{bmatrix}
AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T \\
A^T + A_n + [B_2 D_n C_2]^T & [X_1 B_1 + B_n D_{21}]^T
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
Y_1 & I \\
I & X_1
\end{bmatrix} > 0,
$$

$$
C_1 Y_1 + D_{12} C_n \quad C_1 + D_{12} D_n C_2 \quad Z
$$

$$
D_{11} + D_{12} D_n D_{21} = 0, \quad \text{trace}(Z) < \gamma^2
$$
As before, the controller can be recovered as

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix}
= \begin{bmatrix}
X_2 & X_1 B_2 \\
0 & I
\end{bmatrix}^{-1}
\begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix}
- \begin{bmatrix}
X_1 A Y_1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_2^T & 0 \\
C_2 Y_1 & I
\end{bmatrix}^{-1}
\]

for any full-rank \(X_2\) and \(Y_2\) such that

\[
\begin{bmatrix}
X_1 & X_2 \\
X_2^T & X_3
\end{bmatrix}
= \begin{bmatrix}
Y_1 & Y_2 \\
Y_2^T & Y_3
\end{bmatrix}^{-1}
\]

To find the actual controller, we use the identities:

\[
D_K = (I + D_{K2} D_{22})^{-1} D_{K2}
\]
\[
B_K = B_{K2} (I - D_{22} D_K)
\]
\[
C_K = (I - D_K D_{22}) C_{K2}
\]
\[
A_K = A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K
\]
An LMI for Mixed $H_2-H_\infty$ optimal output feedback control

**Theorem 7.**

The following are equivalent.

- **There exists a** $K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ **such that** $\|S(K, P)\|_{H_2} < \gamma_1$ **and** $\|S(K, P)\|_{H_\infty} < \gamma_2$.

- **There exist** $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ **such that**

$$
\begin{bmatrix}
AY_1 + Y_1A^T + B_2C_n + C_n^TB_2^T & *^T & *^T \\
A^T + A_n + [B_2D_nC_2]^T & X_1A + A^TX_1 + B_nC_2 + C_2^TB_n^T & *^T \\
[B_1 + B_2D_nD_{21}]^T & [X_1B_1 + B_nD_{21}]^T & -I \\
C_1Y_1 + D_{12}C_n & C_1 + D_{12}D_nC_2 & Z
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
Y_1 & I & *^T \\
I & X_1 & *^T \\
C_1Y_1 + D_{12}C_n & C_1 + D_{12}D_nC_2 & Z
\end{bmatrix} > 0,
$$

$$D_{11} + D_{12}D_nD_{21} = 0, \quad \text{trace}(Z) < \gamma_1^2$$

$$
\begin{bmatrix}
AY_1 + Y_1A^T + B_2C_n + C_n^TB_2^T & *^T & *^T & *^T \\
A^T + A_n + [B_2D_nC_2]^T & X_1A + A^TX_1 + B_nC_2 + C_2^TB_n^T & *^T & *^T \\
[B_1 + B_2D_nD_{21}]^T & [X_1B_1 + B_nD_{21}]^T & -\gamma_2I & *^T \\
C_1Y_1 + D_{12}C_n & C_1 + D_{12}D_nC_2 & D_{11} + D_{12}D_nD_{21} & -\gamma_2I
\end{bmatrix} < 0
$$
An LMI for the Kalman Filter! - Continuous Time

System:
\[
\dot{x}(t) = Ax(t) + Bu(t) + w(t)
\]
\[
y(t) = Cx(t) + v(t)
\]

Filter:
\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))
\]
\[
\hat{y}(t) = C\hat{x}(t)
\]

Error:
\[
\dot{e}(t) = (A + LC)e(t) + w(t) + Lv(t)
\]

The Kalman Filter chooses $L$ to minimize the cost $J = E[e^T e]$.
\[
L = \Sigma C^T V_2^{-1}
\]

where $V_1 = E[w(t)w(t)^T]$ and $V_2 = E[v(t)v(t)^T]$ and $\Sigma$ satisfies
\[
A\Sigma + \Sigma A^T + V_1 = \Sigma C^T V_2^{-1} C\Sigma
\]

If we choose $u(t) = K\hat{x}(t)$ where $A + BK$ is stable,
- $A + LC$ is stable if system is observable (not detectable).
- Closed-Loop is stable by the separation principle (has Luenberger form).
- A Dual to the LQR problem. Replace $(A, B, Q, R, K)$ with $(A^T, C^T, V_1, V_2, L^T)$.
Kalman Filter - Discrete Time

Assume the system is driven by noise $w_k$ (no feedback)

$$x_{k+1} = Ax_k + w_k, \quad y_k = Cx_k + v_k$$

The steady-state Kalman filter is an estimator of the form:

$$\hat{x}_{k+1} = A\hat{x}_k + L(C\hat{x}_k - y_k),$$

where $v_k$ is sensor noise. This gives error ($e_k = x_k - \hat{x}_k$) dynamics

$$e_{k+1} = (A + LC)e_k$$

For the Kalman Filter, we choose $L = A\Sigma C^T(C\Sigma C^T + V)^{-1}$ where $V = \mathbf{E}[v_k v_k^T]$ and $\Sigma$ is the steady-state covariance of the error in the estimated state and satisfies

$$\Sigma = A\Sigma A^T + W - A\Sigma C^T(C\Sigma C^T + V)^{-1}C\Sigma A^T$$

where $W = \mathbf{E}[w_k w_k^T]$. For the unsteady Kalman filter, $\Sigma_k$ is updated at each time-step.

- If $(A, W)$ controllable and $(C, A)$ observable, then $A + LC$ is stable.
- Again, dual to discrete-time LQR (which we haven’t solved here!)