Lecture 11: Relationship between $H_2$, LQG and LGR and LMIs for state and output feedback $H_2$ synthesis
To solve the $H_\infty$-optimal state-feedback problem, we solve

$$\min_{\gamma, X_1, Y_1, A_n, B_n, C_n, D_n} \gamma \quad \text{such that}$$

$$
\begin{bmatrix}
X_1 & I \\
I & Y_1
\end{bmatrix} > 0
$$

$$
\begin{bmatrix}
AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\
A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\
[B_1 + B_2 D_n D_21]^T & [X B_1 + B_n D_21]^T & -\gamma I \\
C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I
\end{bmatrix} < 0
$$
Conclusion

Then, we construct our controller using

\[
D_K = (I + D_{K2}D_{22})^{-1}D_{K2}
\]

\[
B_K = B_{K2}(I - D_{22}D_K)
\]

\[
C_K = (I - D_KD_{22})C_{K2}
\]

\[
A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\]

where

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix} = \begin{bmatrix}
X_2 & X_1B_2 \\
0 & I
\end{bmatrix}^{-1} \begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix} - \begin{bmatrix}
X_1AY_1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
Y_2^T & 0 \\
C_2Y_1 & I
\end{bmatrix}^{-1}.
\]

and where \(X_2\) and \(Y_2\) are any matrices which satisfy \(X_2Y_2^T = I - X_1Y_1\).

- e.g. Let \(Y_2 = I\) and \(X_2 = I - X_1Y_1\).
- The optimal controller is NOT uniquely defined.
- Don’t forget to check invertibility of \(I - D_{22}D_K\)
Conclusion

The $H_\infty$-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$

Minimizes the effect of external input ($w$) on external output ($z$).

$$\|z\|_{L_2} \leq \|S(P, K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain
**H₂-optimal control**

Motivation

H₂-optimal control minimizes the H₂-norm of the transfer function.

- The H₂-norm has no direct interpretation.

\[ \|G\|_{H_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(i\omega)^*\hat{G}(i\omega))d\omega \]

Motivation: Assume external input, w, is Gaussian noise with signal variance \( S_w \). Then,

\[ E[w(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{S}_w(i\omega))d\omega \]

**Theorem 1.**

For an LTI system \( P \), if \( w \) is noise with spectral density \( \hat{S}_w(\omega) \) and \( z = Pw \), then \( z \) is noise with density

\[ \hat{S}_z(\omega) = \hat{P}(\omega)\hat{S}(\omega)\hat{P}(\omega)^* \]
Then the output $z = Gw$ has signal variance (Power)

$$E[z(t)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\omega)^* S(\omega) \hat{G}(\omega)) d\omega$$

$$\leq ||S||_{H_\infty} ||G||_{H_2}^2$$

If the input signal is white noise, then $\hat{S}(\omega) = I$ and

$$E[z(t)^2] = ||\hat{G}||_{H_2}^2$$
Now suppose the noise is colored with variance $\hat{S}_w(\omega)$. Now define $\hat{H}$ as $\hat{H}(\omega)\hat{H}(\omega)^* = \hat{S}_w(\omega)$ and the filtered system.

$$\hat{P}_s(s) = \begin{bmatrix} \hat{P}_{11}(s)\hat{H}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s)\hat{H}(s) & \hat{P}_{22}(s) \end{bmatrix}$$

Now, applying feedback to the filtered plant, we get

$$\underline{S}(P_s, K)(s) = P_{11}\hat{H} + P_{12}(I - KP_{22})^{-1}KP_{21}\hat{H} = \underline{S}(P, K)\hat{H}$$

Now the spectral density, $\hat{S}_z$ of the output of the true plant using colored noise equals the output of the artificial plant under white noise, i.e.

$$S_z(s) = \underline{S}(P, K)(s)\hat{S}_w(s)\underline{S}(P, K)(s)^*$$

$$= \underline{S}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\underline{S}(P, K)(s)^* = \hat{S}(P_s, K)(s)\hat{S}(P_s, K)(s)^*$$

Thus if $K$ minimizes the $H_2$-norm of the filtered plant ($\|\hat{S}(P_s, K)\|_{H_2}^2$), it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$. 

Now suppose the noise is colored with variance $\hat{S}_w(\omega)$. Now define $\hat{H}$ as $\hat{H}(\omega)H_0(\omega) = \hat{S}_w(\omega)$ and the filtered system.

\[
\hat{P}(s) = \begin{bmatrix} \hat{P}_{11}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s) & \hat{P}_{22}(s) \end{bmatrix}
\]

Now, applying feedback to the filtered plant, we get

\[
\hat{S}(P, K)(s) = \hat{P}_{11}(s) + \hat{P}_{12}(s)(I - K\hat{P}_{22}(s))^{-1}K\hat{P}_{21}(s)H = \hat{S}(P, K)H
\]

Now the spectral density, $\hat{S}$, of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

\[
\hat{S}(z) = \hat{S}(P, K)(s)\hat{S}(P, K)(s)^* = \hat{S}(P, K)(s)^*\hat{S}(P, K)(s)
\]

Thus if $K$ minimizes the $H_2$-norm of the filtered plant ($\|\hat{S}(P, K)\|^2_{H_2}$), it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$.

Alternatively, we can write

\[
\min_K \| S(P, K)w \|_{var} = \min_K \| S(P, K)Hu \|_{var, u=white} = \min_K \| S(P, K)H \|_{H_2}
\]
Theorem 2.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.
2. There exists some $X > 0$ such that

$$\text{trace } CXC^T < \gamma^2,\quad AX + XA^T + BB^T < 0$$

Recall that the Controllability Grammian is a solution!

- Recall how the proof works.
Proof.

Suppose $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Controllability Grammian is defined as

$$X_c = \int_0^\infty e^{At} BB^T e^{A^T} dt$$

Now recall the Laplace transform

$$\left( \Lambda e^{At} \right) (s) = \int_0^\infty e^{At} e^{-ts} dt$$

$$= \int_0^\infty e^{-(sI-A)t} dt$$

$$= -(sI - A)^{-1} e^{-(sI-A)t} \bigg|_{t=-\infty}^{t=0}$$

Hence $\left( \Lambda Ce^{At} B \right) (s) = C(sI - A)^{-1} B$. 

Proof.

\[
(\Lambda C e^{At} B)(s) = C(sI - A)^{-1} B \implies \\
\| \hat{P} \|_{H_2}^2 = \| C(sI - A)^{-1} B \|_{H_2}^2 \\
= \frac{1}{2\pi} \int_0^\infty \text{Trace}((C(\omega I - A)^{-1} B)^*(C(\omega I - A)^{-1} B)) d\omega \\
= \frac{1}{2\pi} \int_0^\infty \text{Trace}((C(\omega I - A)^{-1} B)(C(\omega I - A)^{-1} B)^*) d\omega \\
= \text{Trace} \int_{-\infty}^\infty C e^{At} B B^* e^{A^*t} C^* dt \\
= \text{Trace} C X_c C^T 
\]

Thus \( X_c \geq 0 \) and \( \text{Trace} C X_c C^T = \| \hat{P} \|_{H_2}^2 < \gamma^2 \).
Proof.

Likewise \( \text{Trace} B^T X_o B = \| \hat{P} \|_{H_2}^2 \) where \( X_o \) is the observability Grammian. To show that we can take strict the inequality \( X > 0 \), we simply let

\[
X = \int_0^\infty e^{At} \left( BB^T + \epsilon I \right) e^{A^T} dt
\]

for sufficiently small \( \epsilon > 0 \). Furthermore, we already know the controllability grammmian \( X_c \) and thus \( X_\epsilon \) satisfies the Lyapunov inequality.

\[
A^T X_\epsilon + X_\epsilon A + BB^T < 0
\]

These steps can be reversed to obtain necessity.
$H_2$-optimal control

Full-State Feedback

Let's consider the full-state feedback problem

\[ \hat{G}(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \]

- $D_{12}$ is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

\[ \hat{K}(s) = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \]
Theorem 3. The following are equivalent.

1. $\|S(K, P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ for some $Z$ and $X > 0$ where

$$
\left[
\begin{array}{cc}
A & B_2
\end{array}
\right]
\left[
\begin{array}{c}
X \\
Z
\end{array}
\right]
+ 
\left[
\begin{array}{cc}
X & Z^T
\end{array}
\right]
\left[
\begin{array}{c}
A^T \\
B_2^T
\end{array}
\right]
+ B_1B_1^T < 0
$$

$$
\text{Trace}\left[
C_1X + D_{12}Z
\right]X^{-1}\left[
C_1X + D_{12}Z
\right] < \gamma^2
$$

However, this is nonlinear, so we need to reformulate using the Schur Complement.
Applying the Schur Complement gives the alternative formulation convenient for control.

**Theorem 4.**

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. $A$ is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.

2. There exists some $X, W > 0$ such that

$$
\begin{bmatrix}
A^T X + XA & XB \\
B^T X & -\gamma I
\end{bmatrix} < 0, \quad
\begin{bmatrix}
X & C^T \\
C & W
\end{bmatrix} > 0, \quad \text{Trace}W < \gamma^2
$$
Theorem 5.

The following are equivalent.

1. $\| S(K, P) \|_{H_2} < \gamma$.

2. $K = ZX^{-1}$ for some $Z$ and $X > 0$ where

$$
\begin{bmatrix}
    A & B_2 \\
    C_1X + D_{12}Z & W
\end{bmatrix}
\begin{bmatrix}
    X \\
    Z
\end{bmatrix}
+ \begin{bmatrix}
    X & Z^T
\end{bmatrix}
\begin{bmatrix}
    A^T \\
    B_2^T
\end{bmatrix}
+ B_1B_1^T < 0
$$

$$
\text{Trace}W < \gamma^2
$$

Thus we can solve the $H_2$-optimal static full-state feedback problem.
$H_2$-optimal control

Relationship to LQR

The LQR Problem:

- **Full-State Feedback**
- **Choose** $K$ **to minimize the cost function**

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

subject to dynamic constraints

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \quad x(0) = x_0$$

Trying to minimize the effect of $x_0$ on a weighted-$L_2$-norm of the regulated output.
To solve the LQR problem using $H_2$ optimal state-feedback control, let

- $C_1 = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix}$
- $D_{12} = \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix}$
- $B_2 = B$ and $B_1 = I$.

So that

$$S(\hat{P}, \hat{K}) = \begin{bmatrix} A + B_2 K & B_1 \\ C_1 + D_{12} K & D_{11} \end{bmatrix} = \begin{bmatrix} A + BK & I \\ \frac{Q^{1/2}}{C_1 + D_{12} K} & 0 \end{bmatrix}$$

And solve the $H_2$ full-state feedback problem. Then if

$$\dot{x}(t) = A_{CL} x(t) = (A + BK) x(t) = Ax(t) + Bu(t)$$
$$u(t) = K x(t), \quad x(0) = x_0$$

Then $x(t) = e^{A_{CL} t} x_0$
**H₂-optimal control**

Relationship to LQR

If

\[
\begin{align*}
\dot{x}(t) &= A_CL x(t) = (A + BK)x(t) = Ax(t) + Bu(t) \\
u(t) &= Kx(t), \quad x(0) = x_0
\end{align*}
\]

then \( x(t) = e^{A_CL t}x_0 \), \( u(t) = Ke^{A_CL t}x_0 \) and

\[
\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt = \int_0^\infty x_0^T e^{A_CL t} (Q + K^T R K) e^{A_CL t} x_0 dt
\]

\[
= \text{Trace} \int_0^\infty x_0^T e^{A_CL t} \begin{bmatrix} Q^{1/2} \\ R^{1/2} K \end{bmatrix}^T \begin{bmatrix} Q^{1/2} \\ R^{1/2} K \end{bmatrix} e^{A_CL t} x_0 dt
\]

\[
= \|x_0\|^2 \text{Trace} \int_0^\infty B_1 e^{A_CL t} (C_1 + D_{12} K)^T (C_1 + D_{12} K) e^{A_CL t} B_1^T dt
\]

\[
= \|x_0\|^2 \|S(K, P)\|_{H_2}^2
\]

Thus LQR reduces to a special case of \( H_2 \) static state-feedback.
Theorem 6 (Scherer, Gahinet).

The following are equivalent.

- There exists a $\hat{K} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ such that $\|S(K, P)\|_{H_2} < \gamma$.

- There exist $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ such that
  \[
  \begin{bmatrix}
  AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T \\
  A^T + A_n + [B_2 D_n C_2]^T \\
  [B_1 + B_2 D_n D_{21}]^T
  \end{bmatrix}
  \begin{bmatrix}
  X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T \\
  [X_1 B_1 + B_n D_{21}]^T \\
  -I
  \end{bmatrix}
  < 0,
  \]

  \[
  \begin{bmatrix}
  Y_1 & I & *^T \\
  I & X_1 & *^T \\
  C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z
  \end{bmatrix}
  \]

  $D_{11} + D_{12} D_n D_{21} = 0$,  

  $\text{trace}(Z) < \gamma^2$
As before, the controller can be recovered as

\[
\begin{bmatrix}
A_{K2} & B_{K2} \\
C_{K2} & D_{K2}
\end{bmatrix} =
\begin{bmatrix}
X_2 & X_1 B_2 \\
0 & I
\end{bmatrix}^{-1}
\begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix} -
\begin{bmatrix}
X_1 A Y_1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_2^T & 0 \\
C_2 Y_1 & I
\end{bmatrix}^{-1}
\]

for any full-rank \(X_2\) and \(Y_2\) such that

\[
\begin{bmatrix}
X_1 & X_2 \\
X_2^T & X_3
\end{bmatrix} =
\begin{bmatrix}
Y_1 & Y_2 \\
Y_2^T & Y_3
\end{bmatrix}^{-1}
\]

To find the actual controller, we use the identities:

\[
D_K = (I + D_{K2} D_{22})^{-1} D_{K2}
\]

\[
B_K = B_{K2}(I - D_{22} D_K)
\]

\[
C_K = (I - D_K D_{22}) C_{K2}
\]

\[
A_K = A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K
\]
The following are equivalent.

1. There exists a $K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ such that $\|S(K, P)\|_{H_2} < \gamma_1$ and $\|S(K, P)\|_{H_\infty} < \gamma_2$.

2. There exist $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ such that

$$
\begin{bmatrix}
AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & B_1 A^T + A_n^T + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & \gamma_2 I \\
A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & \gamma_2 I & 0 \\
[B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & 0 & -I \\
C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z & 0
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & B_1 A^T + A_n^T + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & \gamma_2 I \\
A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & \gamma_2 I & 0 \\
[B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & 0 & -I \\
C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z & 0
\end{bmatrix} < 0.
$$

$$
D_{11} + D_{12} D_n D_{21} = 0, \quad \text{trace}(Z) < \gamma_1^2
$$

M. Peet

Lecture 11: 21 / 23
An LMI for the Kalman Filter! - Continuous Time

System:
\[
\dot{x}(t) = Ax(t) + Bu(t) + w(t) \\
y(t) = Cx(t) + v(t)
\]

Filter:
\[
\dot{x}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)) \\
\hat{y}(t) = C\hat{x}(t)
\]

Error:
\[
\dot{e}(t) = (A + LC)e(t) + w(t) + Lv(t)
\]

The Kalman Filter chooses \( L \) to minimize the cost \( J = \mathbb{E}[e^T e] \).
\[
L = \Sigma C^T V^{-1}_2
\]

where \( V_1 = \mathbb{E}[w(t)w(t)^T] \) and \( V_2 = \mathbb{E}[v(t)v(t)^T] \) and \( \Sigma \) satisfies
\[
A\Sigma + \Sigma A^T + V_1 = \Sigma C^T V^{-1}_2 C\Sigma
\]

If we choose \( u(t) = K\hat{x}(t) \) where \( A + BK \) is stable,

- \( A + LC \) is stable if system is observable (not detectable).
- Closed-Loop is stable by the separation principle (has Luenberger form).
- A Dual to the LQR problem. Replace \((A, B, Q, R, K)\) with \((A^T, C^T, V_1, V_2, L^T)\)
Kalman Filter - Discrete Time

Assume the system is driven by noise \( w_k \) (no feedback)

\[
x_{k+1} = Ax_k + w_k, \quad y_k = Cx_k
\]

The steady-state Kalman filter is an estimator of the form:

\[
\hat{x}_{k+1} = A\hat{x}_k + L(C\hat{x}_k - y_k + v_k),
\]

where \( v_k \) is sensor noise. This gives error \( (e_k = x_k - \hat{x}_k) \) dynamics

\[
e_{k+1} = (A + LC)e_k
\]

For the Kalman Filter, we choose \( L = A\Sigma C^T(C\Sigma C^T + V)^{-1} \) where \( V = E[v_k v_k^T] \) and \( \Sigma \) is the steady-state covariance of the error in the estimated state and satisfies

\[
\Sigma = A\Sigma A^T + W - A\Sigma C^T(C\Sigma C^T + V)^{-1}C\Sigma A^T
\]

where \( W = E[w_k w_k^T] \) For the unsteady Kalman filter, \( \Sigma_k \) is updated at each time-step.

- If \((A, W)\) controllable and \((C, A)\) observable, then \( A + LC \) is stable.
- Again, dual to discrete-time LQR (which we haven’t solved here!)