LMI Methods in Optimal and Robust Control

Matthew M. Peet
Arizona State University

Lecture 13: LMI methods for optimal control and quadratic stability with affine polytopic and interval uncertainty
Types of Uncertainty

We will start with Time-Varying Parametric Uncertainty.

- **Unstructured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathcal{L}(L_2) : \|\Delta\|_{H_\infty} < 1 \} \]

- **Structured, Static, norm-bounded:**
  \[ \Delta := \{ \text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) : |\delta_i| < 1, \bar{\sigma}(\Delta_i) < 1 \} \]

- **Structured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \text{diag}(\Delta_1, \Delta_2, \cdots) \in \mathcal{L}(L_2) : \|\Delta_i\|_{H_\infty} < 1 \} \]

- **Unstructured, Parametric, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1 \} \]

- **Parametric, Polytopic:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \Delta = \sum_i \alpha_i H_i, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \]

- **Parametric, Interval:**
  \[ \Delta := \{ \sum_i \Delta_i \delta_i : \delta_i \in [\delta_i^-, \delta_i^+] \} \]

Each of these can be Time-Varying or Time-Invariant!
Recall the system with Affine Time-Varying uncertainty (No Input).

\[ \dot{x}(t) = (A_0 + \Delta A(t))x(t) \]

where

\[ \Delta A(t) = A_1 \delta_1(t) + \cdots + A_k \delta_k(t) \]

where \( \delta(t) \) lies in either the intervals

\[ \delta_i(t) \in [\delta_i^-, \delta_i^+] \]

or the simplex

\[ \delta(t) \in \{ \delta : \sum_i \alpha_i = 1, \alpha_i \geq 0 \} \]
Definitions: Use Robust Stability for Static Uncertainty

Definition 1.

The system

\[ \dot{x}(t) = (A_0 + \Delta(t))x(t) \]

is Robustly Stable over \( \Delta \) if \( A_0 + \Delta \) is Hurwitz for all \( \Delta \in \Delta \).

Note that Robust Stability DOES NOT imply stability if \( \Delta(t) \) is time-varying.

- It implies that for any \( \Delta \in \Delta \), there exists a \( P(\Delta) > 0 \) such that
  \[
  (A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta
  \]

- For a fixed \( \Delta \), this implies stability using Lyapunov function
  \[
  V(x) = x^T P(\Delta)x.
  \]

- Does not imply stability for TV \( \Delta \) because if \( V(x, t) = x^T P(\Delta(t))x \),
  \[
  \frac{d}{dt} V(x(t), t) = x(t)^T \left( (A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t)) \right) x(t)
  \]
  \[
  + x(t)^T \left( \frac{d}{dt} P(\Delta(t)) \right) x(t)
  \]
  \[
  \leq x(t)^T \left( \frac{d}{dt} P(\Delta(t)) \right) x(t)
  \]
Robust Stability is necessary and sufficient for static uncertainty.


**Definition 2.**

The system

\[ \dot{x}(t) = (A_0 + \Delta(t))x(t) \]

is **Quadratically Stable** over \( \Delta \) if there exists a \( P > 0 \) such that

\[ (A + \Delta)^T P + P(A + \Delta) < 0 \]

for all \( \Delta \in \Delta \).

**Quadratic Stability Implies Stability** of trajectories for any \( \Delta(t) \) with \( \Delta(t) \in \Delta \) for all \( t \geq 0 \).

- Use the Lyapunov function \( V(x) = x^T Px \).

\[
\frac{d}{dt} V(x(t)) = x(t)^T ((A + \Delta(t))^T P + P(A + \Delta(t)))x(t) < 0
\]

**Counterintuitive:**

- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!
**Definition 3.**

The system

\[ \dot{x}(t) = (A_0 + \Delta A(t))x(t) \]

is **Quadratically Stable** over \( \Delta \) if there exists a \( P > 0 \) such that

\[ (A + \Delta A)^T P + P(A + \Delta A) < 0 \quad \text{for all } \Delta A \in \Delta. \]

Quadratic Stability is **CONSERVATIVE**.

- There are Stable Systems which are not Quadratically Stable

\[ \dot{x} = A(t)x, \]

\[ A(t) = \delta_1(t) \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} + \delta_2(t) \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix}, \quad \delta_i \geq 0, \quad \delta_1 + \delta_2 = 1 \]

- Use \( V(x) = \max\{x^T P_1 x, x^T P_2 x\} \) where

\[ P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]
Quadratic Stability is Conservative

Quadratic Stability is sometimes referred to as an “infinite-dimensional LMI”

- Meaning it represents an infinite number of LMI constraints
- One for each possible value of $\Delta \in \Delta$
- Also called a parameterized LMI
- Such LMIs are not tractable.
- For polytopic sets, the LMI can be made finite.
Enforcing an LMI on the entire Polytope

Making an infinite-dimensional LMI finite dimensional

**Theorem 4 (LMIs on the Polytope).**

The following are equivalent for any $H, L_i, R_i$.

\[ H + \sum_i L_i \Delta R_i > 0 \quad \text{for all } \Delta \in Co(\Delta_1, \cdots \Delta_k) \quad (1) \]
\[ H + \sum_i L_i \Delta_j R_i > 0 \quad \text{for all } j = 1, \cdots, k \quad (2) \]

**Proof.**

To show $1 \Rightarrow 2$, note that $\Delta_j \in Co(\Delta_1, \cdots \Delta_k)$ for each $j$. Next, show $2 \Rightarrow 1$.

\[ H + \sum_i L_i \Delta R_i = H + \sum_i L_i \left( \sum_j \alpha_i \Delta_j \right) R_i \quad \alpha_i \geq 0, \sum_j \alpha_j = 1 \]
\[ = \sum_j \alpha_j \left( H + \sum_i L_i \Delta_j R_i \right) \geq 0 \]
**Definition 5.**

The pair \((A + \Delta, \Delta)\) is **Quadratically Stable** over \(\Delta\) if there exists a \(P > 0\) such that

\[
(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta.
\]

**Theorem 6.**

\((A + \Delta, \Delta)\) is quadratically stable over \(\Delta := Co(A_1, \cdots, A_k)\) if and only if there exists a \(P > 0\) such that

\[
(A + A_i)^T P + P(A + A_i) < 0 \quad \text{for } i = 1, \cdots, k
\]

The theorem says the LMI only needs to hold at the **EXTREMAL POINTS** or **VERTICES** of the polytope.

- In Fact, Quadratic Stability MUST be expressed as an LMI
- There is NO Ricatti Eqn. Equivalent.
An LMI for Interval Quadratic Stability

Recall the system with Affine Time-Varying uncertainty.

\[
\dot{x}(t) = (A_0 + \Delta(t))x(t)
\]

where

\[
\Delta(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)
\]

where \(\delta_i(t) \in [-1, 1]\). Note: \(\delta(t)\) lies in the hypercube.

Interval Stability is a Kind of Polytopic Uncertainty.

The vertices of the hypercube define the vertices of the uncertainty set

\[
V := \left\{ A_0 + \sum_i A_i\delta_i, \; \delta_i \in \{-1, 1\} \right\}
\]

**Theorem 7 (Quadratic Stability using \(2^k\) LMI constraints!).**

\((A + \Delta, \Delta)\) is quadratically stable over \(\Delta := Co(V)\) if and only if there exists a \(P > 0\) such that

\[
\left( A_0 + \sum_i A_i\delta_i \right)^T P + P \left( A_0 + \sum_i A_i\delta_i \right) < 0 \quad \text{for every } \delta \in \{-1, 1\}^k
\]
Controller Synthesis is a simple application of the previous theorem:

**Theorem 8.**

There exists a $K$ such that

$$\dot{x}(t) = (A + \Delta_A + (B + \Delta_B)K)x(t)$$

is quadratically stable for $(\Delta_A, \Delta_B) \in Co((A_1, B_2), \cdots, (A_k, B_k))$ if and only if there exists some $P > 0$ and $Z$ such that

$$(A + A_i)P + P(A + A_i)^T + (B + B_i)Z + Z^T(B + B_i)^T < 0 \quad \text{for } i = 1, \cdots k.$$ 

with $K = ZP^{-1}$.

Note that here the controller doesn’t depend on $\Delta$!

- If you want $K$ to depend on $\Delta$, the problem is harder.
- But this would require sensing $\Delta$ in real-time.
Lemma 9 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ and $Z$ such that
\[
\begin{bmatrix}
-rP & AP + BZ \\
(AP + BZ)^T & -rP
\end{bmatrix} + \begin{bmatrix}
0 & A_i P + B_i Z \\
(A_i P + B_i Z)^T & 0
\end{bmatrix} < 0,
\]
\[
AP + BZ + (AP + BZ)^T + A_i P + B_i Z + (A_i P + B_i Z)^T + 2\alpha P < 0,
\]
and
\[
\begin{bmatrix}
AP + BZ + (AP + BZ)^T & c(AP + BZ − (AP + BZ)^T) \\
c((AP + BZ)^T − (AP + BZ)) & AP + BZ + (AP + BZ)^T
\end{bmatrix} + \begin{bmatrix}
A_i P + B_i Z + (A_i P + B_i Z)^T & c(A_i P + B_i Z − (A_i P + B_i Z)^T) \\
c((A_i P + B_i Z)^T − (A_i P + B_i Z)) & A_i P + B_i Z + (A_i P + B_i Z)^T
\end{bmatrix} < 0
\]

for $i = 1, \cdots, k$. Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of $A(\Delta) + B(\Delta)K$ satisfy $|x| \leq r$, $\text{Re } x \leq −\alpha$ and $z + z^* \leq −c|z − z^*|$ for all $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$.  

M. Peet

Lecture 13:
An LMI for Quadratic Polytopic $H_\infty$-Optimal State-Feedback Control

Recall the closed-loop in state feedback is:

$$S(P, K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix}$$

Now add uncertainty to system matrices $A, B_1, B_2, C_1, D_{12}$ and $D_{11}$.

**Theorem 10.**

There exists an $F$ such that $\|S(P(\Delta), K(0, 0, 0, F))\|_{H_\infty} \leq \gamma$ for all $\Delta \in Co(\Delta_1, \cdots \Delta_k)$ if there exist $Y > 0$ and $Z$ such that

$$\begin{bmatrix} Y(A + A_i)^T + (A + A_i)Y + Z^T(B_2 + B_2, i)^T + (B_2 + B_2, i)Z & \ast^T & \ast^T \\ (B_1 + B_1, i)^T & -\gamma I & \ast^T \\ (C_1 + C_1, i)Y + (D_{12} + D_{12, i})Z & D_{11} + D_{11, i} & -\gamma I \end{bmatrix} < 0 \ i = 1, \cdots, k$$

Then $F = ZY^{-1}$.

$$S(P(\Delta), K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix} + \Delta \ \Delta \in Co(\Delta_1, \cdots \Delta_k)$$

$$\Delta_i = \begin{bmatrix} A_1 + B_{2, i} F & B_{1, i} \\ C_{1, i} + D_{12, i} F & D_{11, i} \end{bmatrix}$$
An LMI for Quadratic Polytopic $H_2$-Optimal State-Feedback Control

Similarly

**Theorem 11.**

There exists an $F$ such that $\|S(P(\Delta), K(0, 0, 0, F))\|_{H_2}^2 \leq \gamma$ for all $\Delta \in \mathcal{C}o(\Delta_1, \cdots \Delta_k)$ if there exist $X > 0$ and $Z$ such that

$$
\begin{bmatrix}
AX + B_2Z + XA^T + Z^TB_2^T & B_1 \\
B_1^T & -I
\end{bmatrix} + \begin{bmatrix}
A_iX + B_2,iZ + XA^T_i + Z^TB_2,i & B_1,i \\
B_1,i^T & 0
\end{bmatrix} < 0 \quad i = 1, \ldots, k
$$

$$
\begin{bmatrix}
X \\
(C_1X + D_{12}Z)^T
\end{bmatrix} + \begin{bmatrix}
0 \\
(C_1,iX + D_{12},iZ)^T
\end{bmatrix} > 0 \quad i = 1, \ldots, k
$$

Trace $W < \gamma$

Then $F = ZY^{-1}$.

Similar Steps can be taken for robust estimator design, using the LMIs in Duan.

- However, I am not aware of a robust version of the general optimal output feedback LMI for polytopic uncertainty.
State Equations: \( u_k = Fx_k \)

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
&= Ax_k + BFx_k \\
&= (A + BF)x_k
\end{align*}
\]

Lemma 12.

Suppose there exists some \( X > 0 \) and \( Z \) such that

\[
\begin{bmatrix}
X & AX + BZ \\
(AX + BZ)^T & X
\end{bmatrix} + \begin{bmatrix}
0 & A_iX + B_iZ \\
(A_iX + B_iZ)^T & 0
\end{bmatrix} > 0
\]

then if \( F = ZX^{-1} \), then trajectories of the closed-loop system \( (A + BK) \) are stable for any \( \Delta \in Co(\Delta_1, \cdots \Delta_k) \).