LMI Methods in Optimal and Robust Control

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Lecture 13: LMIs for Optimal Control and Quadratic Stability with Affine Polytopic and Interval Uncertainty
Types of Uncertainty

We will start with Time-Varying Parametric Uncertainty.

- **Unstructured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathcal{L}(L_2) : \|\Delta\|_{H_\infty} < 1 \} \]

- **Structured, Static, norm-bounded:**
  \[ \Delta := \{ \text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots, \Delta_N) : |\delta_i| < 1, \bar{\sigma}(\Delta_i) < 1 \} \]

- **Structured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \text{diag}(\Delta_1, \Delta_2, \cdots) \in \mathcal{L}(L_2) : \|\Delta_i\|_{H_\infty} < 1 \} \]

- **Unstructured, Parametric, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1 \} \]

- **Parametric, Polytopic:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \Delta = \sum_i \alpha_i H_i, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \]

- **Parametric, Interval:**
  \[ \Delta := \{ \sum_i \Delta_i \delta_i : \delta_i \in [\delta_i^-, \delta_i^+] \} \]

Each of these can be Time-Varying or Time-Invariant!
Recall the system with Affine Time-Varying uncertainty (No Input).

\[
\dot{x}(t) = (A_0 + \Delta A(t))x(t)
\]

where

\[
\Delta A(t) = A_1 \delta_1(t) + \cdots + A_k \delta_k(t)
\]

where \( \delta(t) \) lies in either the intervals

\[
\delta_i(t) \in [\delta_i^-, \delta_i^+] 
\]

or the simplex

\[
\delta(t) \in \{ \delta : \sum_i \alpha_i = 1, \alpha_i \geq 0 \}
\]
Definitions: Use Robust Stability for Static Uncertainty

**Definition 1.**
The system
\[ \dot{x}(t) = (A_0 + \Delta(t))x(t) \]
is **Robustly Stable** over \( \Delta \) if \( A_0 + \Delta \) is Hurwitz for all \( \Delta \in \Delta \).

Note that Robust Stability DOES NOT imply stability if \( \Delta(t) \) is time-varying.

- It implies that for any \( \Delta \in \Delta \), there exists a \( P(\Delta) > 0 \) such that
  \[ (A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \]
  for all \( \Delta \in \Delta \).

- For a fixed \( \Delta \), this implies stability using Lyapunov function
  \[ V(x) = x^T P(\Delta)x. \]

- Does not imply stability for TV \( \Delta \) because if \( V(x, t) = x^T P(\Delta(t))x \),
  \[ \frac{d}{dt} V(x(t), t) = x(t)^T \left( (A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t)) \right) x(t) \]
  \[ + x(t)^T \left( \frac{d}{dt} P(\Delta(t)) \right) x(t) \]
  \[ \leq x(t)^T \left( \frac{d}{dt} P(\Delta(t)) \right) x(t) \]
Definitions: Use Robust Stability for Static Uncertainty

**Definition 1.**
The system \( x'(t) = (A_0 + \Delta(t))x(t) \) is **Robustly Stable** over \( \Delta \) if \( A_0 + \Delta \) is Hurwitz for all \( \Delta \in \Delta \).

Note that Robust Stability **DOES NOT** imply stability if \( \Delta(t) \) is time-varying.

- It implies that for any \( \Delta \in \Delta \), there exists a \( P(\Delta) > 0 \) such that \( (A + \Delta)^TP(\Delta) + P(\Delta)(A + \Delta) < 0 \) for all \( \Delta \in \Delta \).
- For a fixed \( \Delta \), this implies stability using Lyapunov function \( V(x) = x^TP(\Delta)x \).
- Does not imply stability for TV \( \Delta \) because if \( V(x,t) = x^TP(\Delta(t))x \),
  \[
  \frac{d}{dt}V(x(t),t) = x(t)^T(A + \Delta(t)^TP(\Delta(t)) + P(\Delta(t))(A + \Delta(t))x(t)
  + x(t)^T(P(\Delta(t))x(t)
  \leq x(t)^T(P(\Delta(t))x(t)
  
Robust Stability is necessary and sufficient for static uncertainty.
Definitions: Quadratic Stability for Dynamic Uncertainty

**Definition 2.**
The system 

\[ \dot{x}(t) = (A_0 + \Delta(t))x(t) \]

is **Quadratically Stable** over \( \Delta \) if there exists a \( P > 0 \) such that

\[ (A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta. \]

Quadratic Stability **Implies Stability** of trajectories for any \( \Delta(t) \) with \( \Delta(t) \in \Delta \) for all \( t \geq 0 \).

- Use the Lyapunov function \( V(x) = x^T P x \).

\[
\frac{d}{dt}V(x(t)) = x(t)^T ((A + \Delta(t))^T P + P(A + \Delta(t))x(t) < 0
\]

**Counterintuitive:**
- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!
Quadratic Stability is Conservative

**Definition 3.**

The system 

\[ \dot{x}(t) = (A_0 + \Delta A(t))x(t) \]

is **Quadratically Stable** over \( \Delta \) if there exists a \( P > 0 \) such that

\[
(A + \Delta A)^T P + P(A + \Delta A) < 0 \quad \text{for all } \Delta A \in \Delta.
\]

Quadratic Stability is **CONSERVATIVE**.

- There are Stable Systems which are not Quadratically Stable

\[
\dot{x} = A(t)x,
\]

\[
A(t) = \delta_1(t) \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} + \delta_2(t) \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix}, \quad \delta_i \geq 0, \quad \delta_1 + \delta_2 = 1
\]

- Use \( V(x) = \max \{ x^T P_1 x, x^T P_2 x \} \) where

\[
P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Quadratic Stability is Conservative

Quadratic Stability is sometimes referred to as an “infinite-dimensional LMI”

- Meaning it represents an infinite number of LMI constraints
- One for each possible value of $\Delta \in \Delta$
- Also called a parameterized LMI
- Such LMI's are not tractable.
- For polytopic sets, the LMI can be made finite.
Enforcing an LMI on the entire Polytope

Making an infinite-dimensional LMI finite dimensional

**Theorem 4 (LMIs on the Polytope).**

The following are equivalent for any $H, L_i, R_i$.

\[
H + \sum_i L_i \Delta R_i > 0 \quad \text{for all } \Delta \in \text{Co}(\Delta_1, \cdots \Delta_k) \tag{1}
\]

\[
H + \sum_i L_i \Delta_j R_i > 0 \quad \text{for all } j = 1, \cdots, k \tag{2}
\]

**Proof.**

To show $1 \Rightarrow 2$, note that $\Delta_j \in \text{Co}(\Delta_1, \cdots \Delta_k)$ for each $j$. Next, show $2 \Rightarrow 1$.

\[
H + \sum_i L_i \Delta R_i = H + \sum_i L_i \left( \sum_j \alpha_i \Delta_j \right) R_i \geq 0
\]

\[
\sum_j \alpha_j \left( H + \sum_i L_i \Delta_j R_i \right) \geq 0
\]
Definition 5.
The pair \((A + \Delta, \Delta)\) is **Quadratically Stable** over \(\Delta\) if there exists a \(P > 0\) such that
\[
(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all} \; \Delta \in \Delta.
\]

Theorem 6.
\((A + \Delta, \Delta)\) is quadratically stable over \(\Delta := \text{Co}(A_1, \cdots, A_k)\) if and only if there exists a \(P > 0\) such that
\[
(A + A_i)^T P + P(A + A_i) < 0 \quad \text{for} \; i = 1, \cdots, k
\]

The theorem says the LMI only needs to hold at the EXTREMAL POINTS or VERTICES of the polytope.
- In Fact, Quadratic Stability MUST be expressed as an LMI
- There is NO Ricatti Eqn. Equivalent.
An LMI for Interval Quadratic Stability

Recall the system with Affine Time-Varying uncertainty.

\[
\dot{x}(t) = (A_0 + \Delta(t))x(t)
\]

where

\[
\Delta(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)
\]

where \(\delta_i(t) \in [-1, 1]\). Note: \(\delta(t)\) lies in the hypercube. Interval Stability is a Kind of Polytopic Uncertainty.

The vertices of the hypercube define the vertices of the uncertainty set

\[
V := \left\{ A_0 + \sum_i A_i\delta_i, \; \delta_i \in \{-1, 1\} \right\}
\]

**Theorem 7 (Quadratic Stability using \(2^k\) LMI constraints!).**

\((A + \Delta, \Delta)\) is quadratically stable over \(\Delta := Co(V)\) if and only if there exists a \(P > 0\) such that

\[
\left( A_0 + \sum_i A_i\delta_i \right)^T P + P \left( A_0 + \sum_i A_i\delta_i \right) < 0 \quad \text{for every } \delta \in \{-1, 1\}^k
\]
Controller Synthesis is a simple application of the previous theorem:

**Theorem 8.**

There exists a $K$ such that

$$\dot{x}(t) = (A + \Delta A + (B + \Delta B)K)x(t)$$

is quadratically stable for $(\Delta A, \Delta B) \in Co((A_1, B_2), \cdots, (A_k, B_k))$ if and only if there exists some $P > 0$ and $Z$ such that

$$(A + A_i)P + P(A + A_i)^T + (B + B_i)Z + Z^T(B + B_i)^T < 0 \quad \text{for } i = 1, \cdots k.$$

with $K = ZP^{-1}$.

Note that here the controller doesn’t depend on $\Delta$!

- If you want $K$ to depend on $\Delta$, the problem is harder.
- But this would require sensing $\Delta$ in real-time.
Lemma 9 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ and $Z$ such that
\[
\begin{bmatrix}
-rP & AP + BZ \\
(AP + BZ)^T & -rP
\end{bmatrix}
+ \begin{bmatrix}
0 & A_iP + B_iZ \\
(A_iP + B_iZ)^T & 0
\end{bmatrix} < 0,
\]
\[
AP + BZ + (AP + BZ)^T + A_iP + B_iZ + (A_iP + B_iZ)^T + 2\alpha P < 0,
\]
and
\[
\begin{bmatrix}
AP + BZ + (AP + BZ)^T & c(AP + BZ - (AP + BZ)^T) \\
(c((AP + BZ)^T - (AP + BZ)) & AP + BZ + (AP + BZ)^T
\end{bmatrix}
+ \begin{bmatrix}
A_iP + B_iZ + (A_iP + B_iZ)^T & c(A_iP + B_iZ - (A_iP + B_iZ)^T) \\
(c((A_iP + B_iZ)^T - (A_iP + B_iZ)) & A_iP + B_iZ + (A_iP + B_iZ)^T
\end{bmatrix} < 0
\]
for $i = 1, \cdots, k$. Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of
\[A(\Delta) + B(\Delta)K\]
satisfy $|x| \leq r$, $\text{Re } x \leq -\alpha$ and $z + z^* \leq -c|z - z^*|$ for all $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$.

Of course, if $\Delta$ is time-varying, eigenvalues are meaningless.
An LMI for Quadratic Polytopic $H_\infty$-Optimal State-Feedback Control

Recall the closed-loop in state feedback is:

$$\mathcal{S}(P, K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix}$$

Now add uncertainty to system matrices $A, B_1, B_2, C_1, D_{12}$ and $D_{11}$.

**Theorem 10.**

There exists an $F$ such that $\|\mathcal{S}(P(\Delta), K(0, 0, 0, F))\|_{H_\infty} \leq \gamma$ for all $\Delta \in \text{Co}(\Delta_1, \cdots, \Delta_k)$ if there exist $Y > 0$ and $Z$ such that

$$\begin{bmatrix} Y(A + A_i)^T + (A + A_i) Y + Z^T (B_2 + B_2, i)^T + (B_2 + B_2, i) Z & *^T \\ (B_1 + B_1, i)^T & *^T \\ (C_1 + C_1, i) Y + (D_{12} + D_{12}, i) Z & -\gamma I \\ & D_{11} + D_{11, i} \end{bmatrix} < 0 \quad i = 1, \cdots, k$$

Then $F = Z Y^{-1}$.

$$\mathcal{S}(P(\Delta), K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix} + \Delta \quad \Delta \in \text{Co}(\Delta_1, \cdots, \Delta_k)$$

$$\Delta_i = \begin{bmatrix} A_i + B_2, i F & B_1, i \\ C_1, i + D_{12, i} F & D_{11, i} \end{bmatrix}$$
In this case, the uncertain system is:

\[
\begin{align*}
\dot{x}(t) &= (A + \Delta A(t))x(t) + (B_1 + \Delta B_1(t))w(t) + (B_2 + \Delta B_2(t))u(t) \\
y(t) &= (C_1 + \Delta C_1(t))x(t) + (D_{11} + \Delta D_{11}(t))w(t) + (D_{12} + \Delta D_{12}(t))u(t)
\end{align*}
\]

where

\[
\begin{align*}
\Delta A(t) &= A_1 \delta_1(t) + \cdots + A_k \delta_k(t) \\
\Delta B_1(t) &= B_{1,1} \delta_1(t) + \cdots + B_{1,k} \delta_k(t) \\
\Delta B_2(t) &= B_{2,1} \delta_1(t) + \cdots + B_{2,k} \delta_k(t) \\
\Delta C_1(t) &= C_{1,1} \delta_1(t) + \cdots + C_{1,k} \delta_k(t) \\
\Delta D_{11}(t) &= D_{11,1} \delta_1(t) + \cdots + D_{11,k} \delta_k(t) \\
\Delta D_{12}(t) &= D_{12,1} \delta_1(t) + \cdots + D_{12,k} \delta_k(t)
\end{align*}
\]
An LMI for Quadratic Polytopic $H_2$-Optimal State-Feedback Control

Similarly

**Theorem 11.**

There exists an $F$ such that $\|S(P(\Delta), K(0, 0, 0, F))\|_{H_2}^2 \leq \gamma$ for all $\Delta \in Co(\Delta_1, \cdots, \Delta_k)$ if there exist $X > 0$ and $Z$ such that

\[
\begin{bmatrix}
AX + B_2Z + XA^T + Z^TB_2^T & B_1 \\
B_1^T & -I
\end{bmatrix} + \begin{bmatrix}
A_iX + B_2,iZ + XA_i^T + Z^TB_2,i^T & B_1,i \\
B_1,i^T & 0
\end{bmatrix} < 0 \quad i = 1, \ldots, k
\]

\[
\begin{bmatrix}
X & (C_1X + D_{12}Z)^T \\
C_1X + D_{12}Z & W
\end{bmatrix} + \begin{bmatrix}
0 & (C_1,iX + D_{12,i}Z)^T \\
C_1,iX + D_{12,i}Z & 0
\end{bmatrix} > 0 \quad i = 1, \ldots, k
\]

$\text{Trace}W < \gamma$

Then $F = ZY^{-1}$.

Similar Steps can be taken for robust estimator design, using the LMIs in Duan.

- However, I am not aware of a robust version of the general optimal output feedback LMI for polytopic uncertainty.
State Equations: Let \( u(k) = Fx(k) \) In this case, the uncertain system is:

\[
x(k + 1) = (A + \Delta A(k)) x(k) + (B + \Delta B(k)) u(k)
= (A + \Delta A(k) + BF + \Delta B(k)F)x(k)
\]

where

\[
\Delta A(k) = A_1 \delta_1(t) + \cdots + A_m \delta_m(k)
\]
\[
\Delta B(k) = B_1 \delta_1(k) + \cdots + B_m \delta_m(k)
\]

Theorem 12.

There exists a \( F \) such that

\[
x_{k+1} = (A + \Delta A + (B + \Delta B)F)x_k
\]
is quadratically stable for \((\Delta A, \Delta B) \in \text{Co}((A_1, B_2), \cdots, (A_k, B_k))\) if and only if there exists some \( X > 0 \) and \( Z \) such that

\[
\begin{bmatrix}
X & AX + BZ \\
(AX + BZ)^T & X
\end{bmatrix} + \begin{bmatrix}
0 & A_iX + B_iZ \\
(A_iX + B_iZ)^T & 0
\end{bmatrix} > 0 \text{ for } i = 1, \cdots m.
\]

In this case, we have \( F = ZP^{-1} \).