LMI Methods in Optimal and Robust Control

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Lecture 14: LMIs for Robust Control in the LFT Framework
Types of Uncertainty

In this Lecture, we will cover

- **Unstructured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathcal{L}(L_2) : \|\Delta\|_{H_\infty} < 1 \} \]

- **Structured, Static, norm-bounded:**
  \[ \Delta := \{ \text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) : |\delta_i| < 1, \bar{\sigma}(\Delta_i) < 1 \} \]

- **Structured, Dynamic, norm-bounded:**
  \[ \Delta := \{ \Delta_1, \Delta_2, \cdots \in \mathcal{L}(L_2) : \|\Delta_i\|_{H_\infty} < 1 \} \]

- **Unstructured, Static, norm-bounded:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1 \} \]

- **Parametric, Polytopic:**
  \[ \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \Delta = \sum_i \alpha_i H_i, \alpha_i \geq 0, \sum_i \alpha_i = 1 \} \]

- **Parametric, Interval:**
  \[ \Delta := \{ \sum_i \Delta_i \delta_i : \delta_i \in [\delta_i^-, \delta_i^+] \} \]

Each of these can be Time-Varying or Time-Invariant!
The interval and polytopic cases rely on **Linearity** of the uncertain parameters.

\[
\dot{x}(t) = (A_0 + \Delta(t))x(t)
\]

The Linear-Fractional Transformation, however

\[
\begin{bmatrix}
\dot{x}_1 \\
z(t)
\end{bmatrix}
= \mathcal{S}(P, \Delta)
\begin{bmatrix}
x_1(t) \\
F(t)
\end{bmatrix}
= (P_{22} + P_{21}(I - \Delta P_{11})^{-1}\Delta P_{12})
\begin{bmatrix}
x(t) \\
F(t)
\end{bmatrix}
\]

is an arbitrary rational function.

We focus on two results:

- The S-Procedure for Unstructured Uncertainty Sets
- The Structured Singular Value for Structured Uncertainty Sets.
Robust Stability

Questions:

- Is $S(\Delta, M)$ stable for all $\Delta \in \Delta$?
- Is $I - \Delta M_{11}$ invertible for all $\Delta \in \Delta$?
Redefine Robust and Quadratic Stability

Suppose we have the system

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]

**Definition 1.**

The pair \((M, \Delta)\) is **Robustly Stable** if \((I - M_{22}\Delta)\) is invertible for all \(\Delta \in \Delta\).

Alternatively, if

\[
\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}
\]

**Definition 2 (Continuous-Time).**

The pair \((M, \Delta)\) is **Robustly Stable** if for some \(\beta > 0\),

\[
M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + \beta I \text{ is Hurwitz for all } \Delta \in \Delta.
\]

Alternatively, if

\[
\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x_k \\ w_k \end{bmatrix}
\]

**Definition 3 (Discrete-Time).**

The pair \((M, \Delta)\) is **Robustly Stable** if

\[
\rho(M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}) = \beta < 1 \text{ for all } \Delta \in \Delta.
\]
Focus on the 1,1 block of $\tilde{S}(M, \Delta)$:
If
$$\dot{x}(t) = \tilde{S}(M, \Delta)x(t),$$

**Definition 4 (Continuous Time).**

The pair $(M, \Delta)$ is Quadratically Stable if there exists a $P > 0$ such that
$$\tilde{S}(M, \Delta)^TP + P\tilde{S}(M, \Delta) < -\beta I \quad \text{for all } \Delta \in \Delta$$

Alternatively, if
$$x_{k+1} = \tilde{S}(M, \Delta)x_k,$$

**Definition 5 (Discrete Time).**

The pair $(M, \Delta)$ is Quadratically Stable if there exists a $P > 0$ such that
$$\tilde{S}(M, \Delta)^TP\tilde{S}(M, \Delta) - P < -\beta I \quad \text{for all } \Delta \in \Delta$$

for all $\Delta \in \Delta$. 
Consider the state-space representation:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + M p(t), \\
p(t) &= \Delta(t) q(t), \\
q(t) &= N x(t) + Q p(t), \\
\Delta(t) &\in \Delta
\end{align*}
\]

- **Parametric, Norm-Bounded Uncertainty:**

\[
\Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \| \Delta \| \leq 1 \}
\]
Parametric, Norm-Bounded Time-Varying Uncertainty

If we close the loop,

\[ \dot{x}(t) = Ax(t) + Mp(t), \quad p(t) = \Delta(t)(Nx(t) + Qp(t)), \]

\[ p(t) = (I - \Delta(t)Q)^{-1}\Delta(t)Nx(t) \]

\[ \dot{x}(t) = (A + M(I - \Delta(t)Q)^{-1}\Delta(t)N)x(t) \]

But this is complicated, so we seek a simpler approach.

\[ V(x) = x^TPx \]

\[ \dot{V}(x) = x(t)^TP(Ax(t) + Mp(t)) + (Ax(t) + Mp(t))^TPx(t) < 0 \]

for all \( p, x \) such that

\[ \|p\| \leq \|Nx + Qp\| \]
Parametric, Norm-Bounded Uncertainty

**Quadratic Stability:** There exists a $P > 0$ such that

$$x^T P (Ax + Mp) + (Ax + Mp)^T Px < 0$$

for all $[x, p] \in \{ x, p : p = \Delta q, q = Nx + Qp, \Delta \in \Delta \}$

---

**Theorem 6.**

The system

$$\dot{x}(t) = Ax(t) + Mp(t), \quad p(t) = \Delta(t)q(t),$$

$$q(t) = Nx(t) + Qp(t), \quad \Delta \in \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1 \}$$

is quadratically stable if and only if there exists some $P > 0$ such that

$$\begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ MT P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} < 0$$

for all $\begin{bmatrix} x \\ p \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ p \end{bmatrix} : \begin{bmatrix} x \\ p \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \leq 0 \right\}$
Parametric, Norm-Bounded Uncertainty

The quadratic stability condition is a conditional LMI

- Positive on a subset of \([x, p]\)
- \([x, p]\) lies in an ellipsoid (a semialgebraic set).
- Enforcing an LMI on a subset is usually hard.
If

\[
\begin{bmatrix}
  x \\
  p
\end{bmatrix}
\begin{bmatrix}
  A^T P + PA & PM \\
  M^T P & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
  p
\end{bmatrix} < 0
\]

for all

\[
\begin{bmatrix}
  x \\
  p
\end{bmatrix} \in \left\{ \begin{bmatrix}
  x \\
  p
\end{bmatrix} : \begin{bmatrix}
  x \\
  p
\end{bmatrix}
\begin{bmatrix}
  -N^T N & -N^T Q \\
  -Q^T N & I - Q^T Q
\end{bmatrix}
\begin{bmatrix}
  x \\
  p
\end{bmatrix} \leq 0 \right\}
\]

then

\[x^T P(Ax + Mp) + (Ax + Mp)^T P x < 0\]

for all \(x, p\) such that

\[\|p\|^2 \leq \|Nx + Qp\|^2\]

Therefore, since \(p = \Delta q\) implies \(\|p\| \leq \|q\|\), we have quadratic stability. The only if direction is similar.
The S-Procedure
A Significant LMI for your Toolbox

Quadratic stability here requires positivity of a matrix on a subset.
• This is Generally a very hard problem
• NP-hard to determine if \( x^T F x \geq 0 \) for all \( x \geq 0 \). (Matrix Copositivity)

S-procedure to the rescue!

The S-procedure asks the question:
• Is \( z^T F z \geq 0 \) for all \( z \in \{ x : x^T G x \geq 0 \} \)?

**Corollary 7 (S-Procedure).**

\[
z^T F z \geq 0 \text{ for all } z \in \{ x : x^T G x \geq 0 \} \text{ if there exists a scalar } \tau \geq 0 \text{ such that } F - \tau G \succeq 0.
\]

The S-procedure is **Necessary** if \( \{ x : x^T G x > 0 \} \) has an interior point.
The system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Mp(t), & p(t) &= \Delta(t)q(t), \\
q(t) &=Nx(t) + Qp(t), & \Delta \in \Delta := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1\}
\end{align*}
\]
is quadratically stable if and only if there exists some \( \mu \geq 0 \) and \( P > 0 \) such that
\[
\begin{bmatrix}
AP + PA^T & PN^T \\
NP & 0
\end{bmatrix} + \mu \begin{bmatrix}
MM^T & MQ^T \\
QM^T & QQ^T - I
\end{bmatrix} < 0
\]

Noting that the LMI can be written as
\[
\begin{bmatrix}
AP + PA^T & PN^T \\
NP & -\mu I
\end{bmatrix} + \mu \begin{bmatrix}
M \\
Q
\end{bmatrix} \begin{bmatrix}
M \\
Q
\end{bmatrix}^T < 0
\]
or
\[
\begin{bmatrix}
AP + PA^T & PN^T & M^T \\
NP & -\mu I & Q^T \\
M & Q & -\frac{1}{\mu} I
\end{bmatrix} < 0
\]
we see that this condition is simply an \( H_\infty \) gain condition on the nominal system \( \| \cdot \|_{H_\infty} < 1 \).
Parametric, Norm-Bounded Uncertainty

Set $\mu = 1$ and we have an LMI for $\| \cdot \|_{H_\infty} < 1$
Necessity of the Small-Gain Condition

This leads to the interesting result:

\[ q \] \quad \frac{\Delta}{p} \quad \frac{M}{q} \]

If \( \Delta := \{ \Delta \in \mathcal{L}(L_2) : \| \Delta \| \leq 1 \} \), then

- \( \bar{S}(P, \Delta) \in H_\infty \) if and only if \( \| P_{11} \|_{H_\infty} < 1 \)
- The small gain condition is necessary and sufficient for stability.
- Quadratic Stability is equivalent to stability.
- Holds for Dynamic and Parametric Uncertainty
  - Does this mean Quadratic and Robust Stability are Equivalent?
Consider Quadratic Stability in Discrete-Time: \( x_{k+1} = S_l(M, \Delta)x_k \).

**Definition 9.**

\((S_l, \Delta)\) is QS if

\[
S_l(M, \Delta)^T P S_l(M, \Delta) - P < 0 \quad \text{for all} \ \Delta \in \Delta
\]

**Theorem 10 (Packard and Doyle).**

Let \( M \in \mathbb{R}^{(n+m) \times (n+m)} \) be given with \( \rho(M_{11}) \leq 1 \) and \( \sigma(M_{22}) < 1 \). Then the following are equivalent.

1. The pair \((M, \Delta = \mathbb{R}^{m \times m})\) is quadratically stable.
2. The pair \((M, \Delta = \mathbb{C}^{m \times m})\) is quadratically stable.
3. The pair \((M, \Delta = \mathbb{C}^{m \times m})\) is robustly stable.
Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

However, we can add controllers:

**Theorem 11.**

The system with \( u(t) = Kx(t) \) and

\[
\begin{align*}
\dot{x}(t) &= A_0x(t) + Bu(t) + Mp(t), \\
p(t) &= \Delta(t)q(t), \\
q(t) &= Nx(t) + Qp(t) + D_{12}u(t), \\
\Delta &\in \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1 \}
\end{align*}
\]

is quadratically stable if and only if there exists some \( \mu \geq 0 \) and \( P > 0 \) such that

\[
\begin{bmatrix}
(A + BK)P + P(A + BK)^T & P(N + D_{12}K)^T \\
(N + D_{12}K)P & 0
\end{bmatrix} + \mu
\begin{bmatrix}
MM^T & MQ^T \\
QM^T & QQ^T - I
\end{bmatrix} < 0
\]

Of course, this is bilinear in \( P \) and \( K \), so we make the change of variables \( Z = KP \).
Theorem 12.

There exists a $K$ such that the system with $u(t) = Kx(t)$

$$
\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \quad p(t) = \Delta(t)q(t),
$$

$$
q(t) = Nx(t) + Qp(t) + D_{12}u(t), \quad \Delta \in \Delta := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1\}
$$

is quadratically stable if and only if there exists some $\mu \geq 0$, $Z$ and $P > 0$ such that

$$
\begin{bmatrix}
AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\
NP + D_{12} Z & 0
\end{bmatrix} + \mu \begin{bmatrix}
MM^T & MQ^T \\
QM^T & QQ^T - I
\end{bmatrix} < 0.
$$

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the $H_{\infty}$ norm.
Theorem 12. There exists a $K$ such that the system with $u(t) = K x(t)$

\[
\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \quad p(t) = \Delta(t)q(t), \quad q(t) = Nx(t) + Qp(t) + Du(t),
\]

$\Delta \in \Delta := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1\}$ is quadratically stable if and only if there exists some $\mu \geq 0$, $Z$ and $P > 0$ such that

\[
\begin{bmatrix}
AP + BZ + Z^T B^T & PZ^T + Z^T D_1^T \\
NP + D_2 Z & -\mu I
\end{bmatrix} +
\begin{bmatrix}
MM^T & MQ^T \\
QM^T & QQ^T - I
\end{bmatrix} < 0.
\]

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the $H_\infty$ norm.
An LMI for $H_{\infty}$-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

**Theorem 13.**

There exists a $K$ such that the system with $u(t) = Kx(t)$

$$
\dot{x}(t) = Ax(t) + Bu(t) + Mp(t) + B_tw(t), \quad p(t) = \Delta(t)q(t),
$$

$$
q(t) = Nx(t) + D_{12}u(t), \quad \Delta \in \Delta := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1\}
$$

$$
y(t) = Cx(t) + D_{22}u(t)
$$

satisfies $\|y\|_{L_2} \leq \gamma \|u\|_{L_2}$ if there exists some $\mu \geq 0$, $Z$ and $P > 0$ such that

$$
\begin{bmatrix}
AP + BZ + PA^T + Z^TB^T + B_2B_2^T + \mu MM^T & (CP + D_{22}Z)^T & PN^T + Z^TD_{12}^T \\
CP + D_{22}Z & -\gamma^2 I & 0 \\
NP + D_{12}Z & 0 & -\mu I
\end{bmatrix} < 0.
$$

Then $K = ZP^{-1}$ is the corresponding controller.
An LMI for $H_\infty$-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

This is from Boyd page 110. I believe it relies on the following alternative to the S-procedure [Xie, 1992], which is similar to Finsler’s Lemma

**Theorem 14.**

The following are equivalent

1. \[ Q + F \Delta E + E^T \Delta F^T > 0 \quad \text{for all } \| \Delta \| < 1 \]

2. There exists some $\epsilon > 0$ such that

\[ Q + \epsilon FF^T + \epsilon^{-1} E^T E > 0 \]

Unfortunately, to put the LMI in the form of 1 requires us to eliminate the pass-through term $Q$. 

In this case, we set $Q = 0$.

**Theorem 13.**

There exists a $K$ such that the system with $u(t) = Kx(t)$

\[
\dot{x}(t) = Ax(t) + Bu(t) + M_1\Delta(t) + B_2w(t),
q(t) = Nx(t) + D_{12}u(t),
\Delta \in \Delta := \{ \Delta \in \mathbb{R}^{n \times n} : \| \Delta \| \leq 1 \}
\]

\[ y(t) = Cx(t) + D_{22}u(t) \]

satisfies

\[ \| y \|_{L_2} \leq \gamma \| u \|_{L_2} \] if there exists some $\mu \geq 0, Z$ and $P > 0$ such that

\[
\begin{bmatrix}
AP + BZ + 2T B^T + R_2 B_2^T + \mu MM^T (C_P + D_2 E^T) P N^T + 2T B_1^T \\
C_P + D_2 E^T & -T^T & 0 & 0 \\
NP + D_2 E^T & 0 & -\mu I
\end{bmatrix} < 0
\]

Then $K = ZP^{-1}$ is the corresponding controller.
For the case of structured parametric uncertainty, we define the structured set

\[ \Delta = \{ \Delta = \text{diag}(\delta_1 I_{n_1}, \ldots, \delta_s I_{n_s}, \Delta_{s+1}, \ldots, \Delta_{s+f}) : \delta_i \in \mathbb{R}, \Delta \in \mathbb{R}^{n_k \times n_k} \} \]

\[ \Delta = \begin{bmatrix} 
\delta_1 I_{n_1} & & \\
& \ddots & \\
& & \delta_s I_{n_s} \\
& & & \Delta_{s+1} \\
& & & & \ddots \\
& & & & & \Delta_{s+f} \end{bmatrix} \]

- \( \delta \) and \( \Delta \) represent unknown parameters.
- \( s \) is the number of scalar parameters.
- \( f \) is the number of matrix parameters.
The Structured Singular Value

For the case of structured parametric uncertainty, we define the structured singular value.

Definition 15.

Given system $M \in \mathcal{L}(L_2)$ and set $\Delta$ as above, we define the Structured Singular Value of $(M, \Delta)$ as

$$
\mu(M, \Delta) = \inf_{\Delta \in \Delta} \frac{1}{\| I - M\Delta \| \text{ is singular}}
$$

Of course, $\bar{S}(M, \Delta)$ is stable if and only if $\mu(M_{11}, \Delta) < 1$.

- Obviously, $\mu(M, \Delta) < \| M \|$.
- For $\Delta := \{ \Delta \in \mathcal{L}(L_2) : \| \Delta \| \leq 1 \}$, $\mu(M, \Delta) = \| M \|$.
- $\mu(\alpha M, \Delta) = |\alpha| \mu(M, \Delta)$.
- Can increase $M$ by a factor $\frac{1}{\mu(M, \Delta)}$ before losing stability.
- In general, computing $\mu$ is NP-hard unless uncertainty is unstructured.
Suppose $\Theta = \{\Theta : \Theta \Delta = \Delta \Theta \text{ for all } \Delta \in \Delta\}$

- Then $\mu(M, \Delta) = \inf_{\Theta \in \Theta} \|\Theta M \Theta^{-1}\|$.
- $\Theta$ is the set of scalings.
\[ \Delta = \{ \Delta = \text{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) : \delta_i \in \mathbb{R}, \Delta \in \mathbb{R}^{n_k \times n_k} \} \]

Define the set of scalings

\[ P\Theta := \{ \text{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1} I, \cdots, \theta_{s+f} I : \Theta_i > 0, \theta_j > 0 \} \]

**Theorem 16.**

Suppose system \( M \) has transfer function \( \hat{M}(s) = C(sI - A)^{-1}B + D \) with \( \hat{M} \in H_\infty \). The following are equivalent

- There exists \( \Theta \in \Theta \) such that \( \| \Theta M \Theta^{-1} \|^2 < \gamma \).
- There exists \( \Theta \in P\Theta \) and \( X > 0 \) such that

\[
\begin{bmatrix}
A^T X + X A & X B \\
B^T X & -\Theta
\end{bmatrix}
+ \frac{1}{\gamma^2}
\begin{bmatrix}
C^T \\
D^T
\end{bmatrix}
\Theta
\begin{bmatrix}
C & D
\end{bmatrix}
< 0
\]

**Note:** To minimize \( \gamma \), you must use bisection.
An LMI for Stability of Structured, Norm-Bounded Uncertainty

This allows us to generalize the S-procedure to structured uncertainty

**Theorem 17.**

The system

\[
\dot{x}(t) = Ax(t) + Mp(t), \quad p(t) = \Delta(t)q(t),
\]

\[
q(t) = Nx(t) + Qp(t), \quad \Delta \in \Delta, \|\Delta\| \leq 1
\]

is quadratically stable if and only if there exists some \( \Theta \in P\Theta \) and \( P > 0 \) such that

\[
\begin{bmatrix}
AP + PA^T & PN^T \\
NP & 0
\end{bmatrix} + \begin{bmatrix}
M\Theta M^T & M\Theta Q^T \\
Q\Theta M^T & Q\Theta Q^T - \Theta
\end{bmatrix} < 0
\]

This is an LMI in \( \Theta \) and \( P \).

- The constraint \( \Theta \in P\Theta \) is linear

\[
P\Theta := \{\text{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1}I, \cdots, \theta_{s+f}I) : \Theta_i > 0, \theta_j > 0\}\]
An LMI for Stability with Structured, Norm-Bounded Uncertainty

To prove the theorem, we can take a closer look at the scalings:

Since $T\Delta = \Delta T$ for $T \in \Theta$, the system can equivalently be written as

$$\dot{x}(t) = Ax(t) + MT^{-1}p(t), \quad p(t) = \Delta(t)q(t),$$
$$q(t) = TNx(t) + TQT^{-1}p(t), \quad \Delta \in \Delta, \; \|\Delta\| \leq 1$$

for any $T \in \Theta$. Then

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0$$

becomes

$$\begin{bmatrix} AP + PA^T & PN^TT^T \\ TNP & 0 \end{bmatrix} + \begin{bmatrix} MT^{-2}M^T & MT^{-2}QTT^T \\ TQT^{-2}M^T & TQT^{-2}QTT^T - I \end{bmatrix} < 0$$

Pre- and Post-multiplying by

$$\begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix},$$

and using $\Theta = T^{-2} \in P\Theta$, we recover the LMI condition.
An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

**Theorem 18.**

There exists a $K$ such that the system with $u(t) = Kx(t)$

\[
\dot{x}(t) = Ax(t) + Bu(t) + Mp(t), \quad p(t) = \Delta(t)q(t), \\
q(t) = Nx(t) + Qp(t) + D_{12}u(t), \quad \Delta \in \Delta, \quad \|\Delta\| \leq 1
\]

is quadratically stable if and only if there exists some $\Theta \in P\Theta$, $P > 0$ and $Z$ such that

\[
\begin{bmatrix}
AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\
NP + D_{12}Z & 0
\end{bmatrix}
+ \begin{bmatrix}
M\Theta M^T & M\Theta Q^T \\
Q\Theta M^T & Q\Theta Q^T - \Theta
\end{bmatrix} < 0.
\]

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the $H_\infty$ norm.
An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Theorem 18. There exists a $K$ such that the system with $u(t) = Kx(t)$

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Mp(t), \\
p(t) &= \Delta(t)q(t), \\
\Delta \in \Delta, \quad \|\Delta\| \leq 1
\end{align*}
$$

is quadratically stable if and only if there exists some $\Theta \in \mathcal{P}\Theta$, $P > 0$ and $Z$ such that

$$
\begin{bmatrix}
AP + BZ + PA^T + Z^T B^T & PS + Z^T D^T \\
NP + DZ & 0
\end{bmatrix} + \begin{bmatrix}
MPM^T & MQM^T \\
QM^T & QM^2 + 4n
\end{bmatrix} < 0
$$

Then $K = ZP^{-1}$ is a quadratically stabilizing controller.

We can also extend this result to optimal control in the $H_\infty$ norm.
In this case, we set $Q = 0$.

**Theorem 19.**

There exists a $K$ such that the system with $u(t) = K x(t)$

\[
\dot{x}(t) = Ax(t) + Bu(t) + Mp(t) + B_2 w(t), \quad p(t) = \Delta(t) q(t),
\]

\[
q(t) = Nx(t) + D_{12} u(t), \quad \Delta \in \Delta, \quad \|\Delta\| \leq 1
\]

\[
y(t) = C x(t) + D_{22} u(t)
\]

satisfies $\|y\|_{L_2} \leq \gamma \|u\|_{L_2}$ if there exists some $\Theta \in \mathbf{P\Theta}$, $Z$ and $P > 0$ such that

\[
\begin{bmatrix}
AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + M\Theta M^T & (CP + D_{22} Z)^T P N^T + Z^T D_{12}^T \\
CP + D_{22} Z & -\gamma^2 I \\
NP + D_{12} Z & 0
\end{bmatrix} < 0.
\]

Then $K = Z P^{-1}$ is the corresponding controller.
An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Using the equivalent scaled system

\[
\dot{x}(t) = Ax(t) + Bu(t) + MT^{-1}p(t) + B_2w(t), \quad p(t) = \Delta(t)q(t),
\]

\[
q(t) = TNx(t) + TD_{12}u(t), \quad \Delta \in \Delta, \quad \|\Delta\| \leq 1
\]

\[
y(t) = Cx(t) + D_{22}u(t)
\]

we get

\[
\begin{bmatrix}
AP + BZ + P A^T + Z^T B^T + B_2B_2^T + MT^{-2}M^T & (CP + D_{22}Z)^T & P N^T T^T + Z^T D_{12}^T T^T \\
CP + D_{22}Z & -\gamma^2 I & 0 \\
TN + TD_{12}Z & 0 & -I
\end{bmatrix} < 0.
\]

Pre- and Post-multiplying by

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & T^{-1}
\end{bmatrix}, \quad \text{and using } \Theta = T^{-2} \in \mathbf{P}\Theta, \text{ we}
\]

recover the LMI condition.
An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

This is not from Boyd, but should be
Output-Feedback Robust Controller Synthesis

How to Solve the Output Feedback Case???

\[
\inf_K \sup_{\Delta \in \Delta} \| S(\bar{S}(G, \Delta), K) \|_{H_{\infty}}
\]
D-K Iteration

A Heuristic for Dynamic Output Feedback Synthesis

Finally, we mention a Heuristic for Output-Feedback Controller synthesis.

Initialize: $\Theta = I$.

Define:

$$\hat{G}_\Theta(s) = \begin{bmatrix} A & B_1 \Theta^{-\frac{1}{2}} & B_2 \\ \Theta^\frac{1}{2} C_1 & \Theta^\frac{1}{2} D_{11} \Theta^{-\frac{1}{2}} & \Theta^\frac{1}{2} D_{12} \\ C_2 & D_{21} \Theta^{-\frac{1}{2}} & 0 \end{bmatrix}$$

Step 1: Fix $\Theta$ and solve

$$\inf_K \| S(G_\Theta, K) \|_{H_\infty}$$

Step 2: Fix $K$ and minimize $\gamma$ such that there exists $\Theta \in P\Theta$ (or $\Theta \in P\Theta \times I$ if you include the regulated output channel.) and $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} \\ B_{cl}^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix} \Theta \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

where $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ define $S(G_I, K)$. (Requires Bisection).

Step 3: GOTO Step 1
The D-K iteration outlined in this lecture is only valid for *Dynamic Uncertainty*: \( \Delta(t) \).

- Our Scalings \( \Theta \) are time-invariant.

For Static uncertainties, we should search for *Dynamic Scaling Factors*

- \( \Theta(s) \) is a *Transfer Function*
- This is much harder to represent as an LMI (Or by any other method!).
- Matlab has built-in functionality, but it is hard to use.

We will return to \( \mu \) analysis for static uncertainties when we consider more advanced forms of optimization.