LMI Methods in Optimal and Robust Control

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Thanks to S. Lall and P. Parrilo for guidance and supporting material

Lecture 16: Optimization of Polynomials and an LMI for Global Lyapunov Stability
Optimization of Polynomials:
As Opposed to Polynomial Programming

**Polynomial Programming (NOT CONVEX):** $n$ decision variables

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$g_i(x) \geq 0$$

- $f$ and $g_i$ must be convex for the problem to be convex.

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**Optimization of Polynomials:** Lifting to a higher-dimensional space

$$\max_{g, \gamma} \quad \gamma$$

$$f(x) - \gamma = g(x) \quad \text{for all} \quad x \in \mathbb{R}^n$$

$$g(x) \geq 0 \quad \text{for all} \quad x \in \{x \in \mathbb{R}^n : h(x) \geq 0\}$$

- The decision variables are functions (e.g. $g$)
  - One constraint for every possible value of $x$.
- But how to parameterize functions????
- How to enforce an infinite number of constraints???
- **Advantage:** Problem is convex, even if $f, g, h$ are not convex.
Optimization of Polynomials:
Some Examples: Matrix Copositivity

Of course, you already know some applications of Optimization of Polynomials

• Global Stability of Nonlinear Systems

\[ V(x) > \epsilon x^2 \quad \text{for all } y \in \mathbb{R}^n \]
\[ \nabla V(x)^T f(x) < 0 \quad \text{for all } y \in \mathbb{R}^n \]

Stability of Systems with Positive States: Not all states can be negative...

• Cell Populations/Concentrations
• Volume/Mass/Length

We want:

\[ V(x) = x^T Px \geq 0 \quad \text{for all } x \geq 0 \]
\[ \dot{V}(x) = x^T (A^T P + PA)x \leq 0 \quad \text{for all } x \geq 0 \]

• Matrix Copositivity (An NP-hard Problem)

Verify:

\[ x^T Px \geq 0 \quad \text{for all } x \geq 0 \]
Recall: Systems with Uncertainty

\[ \dot{x}(t) = A(\delta)x(t) + B_1(\delta)w(t) + B_2(\delta)u(t) \]
\[ y(t) = C(\delta)x(t) + D_{12}(\delta)u(t) + D_{11}(\delta)w(t) \]

**Theorem 1.**

There exists an \( F(\delta) \) such that \( \|S(P(\delta), K(0, 0, 0, F(\delta)))\|_{H_\infty} \leq \gamma \) for all \( \delta \in \Delta \) if there exist \( Y > 0 \) and \( Z(\delta) \) such that

\[
\begin{bmatrix}
 YA(\delta)^T + A(\delta)Y + Z(\delta)^TB_2(\delta)^T + B_2(\delta)Z(\delta) & T & T \\
 B_1(\delta)^T & -\gamma I & T \\
 C_1(\delta)Y + D_{12}(\delta)Z(\delta) & D_{11}(\delta) & -\gamma I
\end{bmatrix} < 0 \text{ for all } \delta \in \Delta
\]

Then \( F(\delta) = Z(\delta)Y^{-1} \).
The Structured Singular Value, $\mu$

**Definition 2.**

Given system $M \in \mathcal{L}(L_2)$ and set $\Delta$ as above, we define the **Structured Singular Value** of $(M, \Delta)$ as

$$\mu(M, \Delta) = \inf_{\Delta \in \Delta \atop I - M\Delta \text{ is singular}} \frac{1}{||\Delta||}$$

The system

$$\dot{x}(t) = A_0 x(t) + Mq(t), \quad q(t) = \Delta(t)p(t),$$

$$p(t) = Nx(t) + Qq(t), \quad \Delta \in \Delta$$

**Lower Bound for $\mu$:** $\mu \geq \gamma$ if there exists a $P(\delta)$ such that

$$P(\delta) \geq 0 \quad \text{for all} \quad \delta$$

$$\dot{V} = x^T P(\delta)(A_0 x + Mq) + (A_0 x + Mq)^T P(\delta)x < \epsilon x^T x$$

for all $x, q, \delta$ such that

$$(x, q, \delta) \in \left\{ x, q, \delta : q = \text{diag}(\delta_i)(Nx + Qq), \sum_i \delta_i^2 \leq \gamma^2 \right\}$$
Overview

In this lecture, we will show how the LMI framework can be expanded dramatically to other forms of control problems.

1. Positivity of Polynomials
   1.1 Sum-of-Squares

2. Positivity of Polynomials on Semialgebraic sets
   2.1 Inference and Cones
   2.2 Positivstellensatz

3. Applications
   3.1 Nonlinear Analysis
   3.2 Robust Analysis and Synthesis
   3.3 Global optimization
A Generic Convex Optimization Problem:

$$\max_x bx$$

subject to \( Ax \in C \)

The problem is \textit{convex optimization} if

- \( C \) is a convex cone.
- \( b \) and \( A \) are affine.

\textbf{Computational Tractability:} Convex Optimization over \( C \) is tractable if

- The set membership test for \( y \in C \) is in \( P \) (polynomial-time verifiable).
- The variable \( x \) is a finite dimensional vector (e.g. \( \mathbb{R}^n \)).
Optimization of Polynomials is Convex
The variables are finite-dimensional (if we bound the degree)

**Convex Optimization of Functions:** Variables $V \in C[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

\[
\begin{align*}
\max_{V, \gamma} & \quad \gamma \\
\text{subject to} & \quad V(x) - x^T x \geq 0 \quad \forall x \\
& \quad \nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x
\end{align*}
\]

$V$ is the decision variable (infinite-dimensional)

- How to make it finite-dimensional???

The set of polynomials is an infinite-dimensional (but *Countable*) vector space.

- It is **Finite Dimensional** if we bound the degree
- All finite-dimensional vector spaces are equivalent!

But we need a way to parameterize this space...
A Parametrization consists of a **basis** and a set of **parameters** (coordinates)

- The set of polynomials is an infinite-dimensional vector space.
- It is **Finite Dimensional** if we bound the degree
  - The monomials are a simple basis for the space of polynomials

**Definition 3.**
Define $Z_d(x)$ to be the vector of monomial bases of degree $d$ or less.

E.g., if $x \in \mathbb{R}^2$, then the vector of basis functions is

$$Z_2(x_1, x_2)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

and

$$Z_4(x_1)^T = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \end{bmatrix}$$

**Linear Representation**
- Any polynomial of degree $d$ can be represented with a vector $c \in \mathbb{R}^m$
  $$p(x) = c^T Z_d(x)$$
- $c$ is the vector of **parameters** (decision variables).
- $Z_d(x)$ doesn’t change (fixed).

$$2x_1^2 + 6x_1x_2 + 4x_2 + 1 = \begin{bmatrix} 1 & 0 & 4 & 6 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}^T$$
Optimization of Polynomials is Convex

The variables are finite-dimensional (if we bound the degree)

**Convex Optimization of Functions:** Variables $V \in \mathbb{R}[x]$ and $\gamma \in \mathbb{R}$

$$\max_{V, \gamma} \gamma$$

subject to

$$V(x) - x^T x \geq 0 \quad \forall x$$

$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \quad \forall x$$

Now use the polynomial parametrization $V(x) = c^T Z(x)$

• Now $c$ is the decision variable.

**Convex Optimization of Polynomials:** Variables $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$

$$\max_{c, \gamma} \gamma$$

subject to

$$c^T Z(x) - x^T x \geq 0 \quad \forall x$$

$$c^T \nabla Z(x) f(x) + \gamma x^T x \leq 0 \quad \forall x$$
Can LMIs be used for Optimization of Polynomials???

Optimization of Polynomials is NP-Hard!!!

**Problem:** Use a finite number of variables:

$$\max b^T x$$

subject to $A_0(y) + \sum_{i} x_i A_i(y) \succeq 0 \quad \forall y$

The $A_i$ are matrices of polynomials in $y$. e.g. Using multi-index notation,

$$A_i(y) = \sum_{\alpha} A_{i,\alpha} y^\alpha$$

The FEASIBILITY TEST is Computationally Intractable

The problem: “Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?” (i.e. “$p \in \mathbb{R}^+[x]$”) is NP-hard.
How to Find Lyapunov Functions (LF)?

We know a LF by its Properties

What makes a LF a Lyapunov Function?

**Property 1:** Positivity
- The Lyapunov function must be positive (metrics are positive).
- Also \( V(0) = 0 \) if 0 is an equilibrium.

**Property 2:** Negativity along Trajectories
- A Lyapunov Function decreases monotonically in time.
  - A longer trajectory is always “bigger” than a shorter one.
  - As time progresses, the trajectory gets shorter.
  - Hence the LF is always decreasing.

**Thus:** If a function has properties 1) and 2), it is a Lyapunov Function

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**Note:** For Linear Systems, we can restrict the search to quadratic LFs.

**Property 3:** Quadratic (e.g. \( x^T P x \))
- If the system is Linear, the solution map is Linear
- Then if the metric is Quadratic, the Lyapunov function is Quadratic.
  - The Composition of Linear and Quadratic Functions is Quadratic
Lyapunov Functions for Linear ODEs

\[ \dot{x}(t) = Ax(t) \]

if \( x \in \mathbb{R}^n \), then any quadratic function has the form:

\[
V(x) = x^T P x \geq 0
\]

**P3 - Quadratic:** The 15 variables \( P_{ij} \) parameterize all possible quadratic functions on \( x \in \mathbb{R}^5 \).

**P1 - Positive:** \( V(x) > 0 \) for all \( x \neq 0 \) if and only if \( P \) has all positive eigenvalues.
Lyapunov Functions for Nonlinear ODEs

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{bmatrix} = f(x(t), y(t))
\]

Let's try a quadratic function of the form:

\[
V(x, y) = \begin{bmatrix}
x \\
y \\
x y \\
x^2 \\
y^2
\end{bmatrix}^T
\begin{bmatrix}
P_{11} & P_{21} & P_{31} & P_{41} & P_{51} \\
P_{21} & P_{22} & P_{32} & P_{42} & P_{52} \\
P_{31} & P_{32} & P_{33} & P_{43} & P_{53} \\
P_{41} & P_{42} & P_{43} & P_{44} & P_{54} \\
P_{51} & P_{52} & P_{53} & P_{54} & P_{55}
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
x y \\
x^2 \\
y^2
\end{bmatrix} \geq 0
\]

**P3 - Pseudo-Quadratic:** The 15 variables $P_{ij}$ parameterize positive polynomial functions $x, y \in \mathbb{R}^2$ of degree $d \leq 4$.

**P1 - Positive:** $V(x) > 0$ for all $x \neq 0$ if $P$ has all positive eigenvalues.
Can LMIs be used to Optimize Positive Polynomials???
Show Me the LMI!

**Basic Idea:** If there exists a Positive Matrix $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

Positive Matrices ($P \geq 0$) have square roots!

$$P = Q^T Q$$

Hence

$$V(x) = Z_d(x)^T Q^T Q Z_d(x) = (Q Z_d(x))^T (Q Z_d(x))$$
$$= h(x)^T h(x) \geq 0$$

**Conclusion:**

$$V(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n$$

if there exists a $P \geq 0$ such that

$$V(x) = Z_d(x)^T P Z_d(x)$$

• Such a function is called **Sum-of-Squares** (SOS), denoted $V \in \Sigma_s$.
• This is an LMI! Equality constraints relate the coefficients of $V$ (decision variables) to the elements of $P$ (more decision variables).
How Hard is it to Determine Positivity of a Polynomial???

Certificates

**Definition 4.**
A Polynomial, \( f \), is called Positive SemiDefinite (PSD) if

\[
f(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n
\]

**The Primary Problem:** How to enforce the constraint \( f(x) \geq 0 \) for all \( x \)?

**Easy Proof:** Certificate of Infeasibility
- A Proof that \( f \) is NOT PSD.
- i.e. To show that
  \[
f(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n
\]
  is FALSE, we need only find a point \( x \) with \( f(x) < 0 \).

**Complicated Proof:** It is much harder to identify a **Certificate of Feasibility**
- A Proof that \( f \) is PSD.
Question: How does one prove that $f(x)$ is positive semidefinite?

What Kind of Functions do we Know are PSD?

- Any squared function is positive.
- The sum of squared forms is PSD
- The product of squared forms is PSD
- The ratio of squared forms is PSD

So $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ if

$$V(x) = \prod_k \frac{\sum_i f_{ik}(x)^2}{\sum_j h_{jk}(x)^2}.$$ 

But is any PSD polynomial the sum, product, or ratio of squared polynomials?

- An old Question....
A polynomial, \( p(x) \in \mathbb{R}[x] \) is a **Sum-of-Squares (SOS)**, denoted \( p \in \Sigma_s \) if there exist polynomials \( g_i(x) \in \mathbb{R}[x] \) such that

\[
p(x) = \sum_{i=1}^{k} g_i(x)^2.
\]

David Hilbert created a famous list of 23 then-unsolved mathematical problems in 1900.

- Only 10 have been fully resolved.
- The 17th problem has been resolved.

“Given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions?” - D. Hilbert, 1900
Hilbert’s 17th was resolved in the affirmative by E. Artin in 1927.

- Any PSD polynomial is the sum, product and ratio of squared polynomials.
- If $p(x) \geq 0$ for all $x \in \mathbb{R}^n$, then
  \[ p(x) = \frac{g(x)}{h(x)} \]
  where $g, h \in \Sigma_s$.
- If $p$ is positive definite, then we can assume $h(x) = (\sum_i x_i^2)^d$ for some $d$. That is,
  \[ (x_1^2 + \cdots + x_n^2)^d p(x) \in \Sigma_s \]
- If we can’t find a SOS representation (certificate) for $p(x)$, we can try $(\sum_i x_i^2)^d p(x)$ for higher powers of $d$.

Of course this doesn’t answer the question of how we find SOS representations.
Quadratic Parameterization of Polynomials

Quadratic Representation

• Alternative to Linear Parametrization, a polynomial of degree $d$ can be represented by a matrix $M \in \mathbb{S}^m$ as

$$p(x) = Z_d(x)^T M Z_d(x)$$

• However, now the problem may be under-determined

$$\begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}$$

$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 xy^3 + M_6 y^4$$

Thus, there are infinitely many quadratic representations of $p$. For the polynomial

$$f(x) = 4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4,$$

we can use the alternative solution

$$4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4$$

$$= M_1 x^4 + 2M_2 x^3 y + (2M_3 + M_4) x^2 y^2 + 2M_5 xy^3 + M_6 y^4$$
Polynomial Representation - Quadratic

For the polynomial

\[ f(x) = 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4, \]

we require

\[ 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = M_1x^4 + 2M_2x^3y + (2M_3 + M_4)x^2y^2 + 2M_5xy^3 + M_6y^4 \]

Constraint Format:

\[ M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad 10 = M_6. \]

An underdetermined system of linear equations (6 variables, 5 equations).

• This yields a family of quadratic representations, parameterized by \( \lambda \) as

\[ 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -\lambda \\ 2 & -7 + 2\lambda & -1 \\ -\lambda & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \]

which holds for any \( \lambda \in \mathbb{R} \).
Positive Matrix Representation of SOS

Sufficiency

Quadratic Form:

\[ p(x) = Z_d(x)^T M Z_d(x) \]

Consider the case where the matrix \( M \) is positive semidefinite.

Suppose: \( p(x) = Z_d(x)^T M Z_d(x) \) where \( M > 0 \).

- Any positive semidefinite matrix, \( M \geq 0 \) has a square root \( M = PP^T \)

Hence

\[ p(x) = Z_d(x)^T M Z_d(x) = Z_d(x)^T P P^T Z_d(x). \]

Which yields

\[ p(x) = \sum_i \left( \sum_j P_{i,j} Z_{d,j}(x) \right)^2 \]

which makes \( p \in \Sigma_s \) an SOS polynomial.
Moreover: Any SOS polynomial has a quadratic rep. with a PSD matrix.

Suppose: \( p(x) = \sum_i g_i(x)^2 \) is degree \( 2d \) (\( g_i \) are degree \( d \)).

- Each \( g_i(x) \) has a linear representation in the monomials.
  \[ g_i(x) = c_i^T Z_d(x) \]

- Hence
  \[
p(x) = \sum_i g_i(x)^2 = \sum_i Z_d(x)c_i c_i^T Z_d(x) = Z_d(x) \left( \sum_i c_i c_i^T \right) Z_d(x)
  \]

- Each matrix \( c_i c_i^T \geq 0 \). Hence \( Q = \sum_i c_i c_i^T \geq 0 \).
- We conclude that if \( p \in \Sigma_s \), there is a \( Q \geq 0 \) with \( p(x) = Z_d(x)QZ_d(x) \).

**Lemma 6.**

Suppose \( M \) is polynomial of degree \( 2d \). \( M \in \Sigma_s \) if and only if there exists some \( Q \geq 0 \) such that

\[
M(x) = Z_d(x)^T Q Z_d(x).
\]
Thus we can express the search for a SOS certificate of positivity as an LMI.

Take the numerical example

$$4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4$$

The question of an SOS representation is equivalent to

Find

$$M = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_2 & M_4 & M_5 \\ M_3 & M_5 & M_6 \end{bmatrix} \succeq 0$$

such that

$$M_1 = 4; \quad 2M_2 = 4; \quad 2M_3 + M_4 = -7; \quad 2M_5 = -2; \quad M_6 = 10.$$ 

In fact, this is feasible for

$$M = \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$
We can use this solution to construct an SOS certificate of positivity.

\[ 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 4 & 2 & -6 \\ 2 & 5 & -1 \\ -6 & -1 & 10 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \]

\[ = \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x^2 \\ xy \\ y^2 \end{bmatrix} \]

\[ = \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix}^T \begin{bmatrix} 2xy + y^2 \\ 2x^2 + xy + 3y^2 \end{bmatrix} \]

\[ = (2xy + y^2)^2 + (2x^2 + xy + 3y^2)^2 \]
Solving Sum-of-Squares using SDP

Quadratic vs. Linear Representation

**Quadratic Representation:** (Using Matrix $M \in \mathbb{R}^{p \times p}$):

$$p(x) = Z_d(x)^T M Z_d(x)$$

**Linear Representation:** (Using Vector $c \in \mathbb{R}^q$)

$$q(x) = c^T Z_{2d}(x)$$

To constrain $p(x) = q(x)$, we write $[Z_d]_i = x^{\alpha_i}$, $[Z_{2d}]_j = x^{\beta_j}$ and reformulate

$$p(x) = Z_d(x)^T M Z_d(x) = \sum_{i,j} M_{i,j} x^{\alpha_i + \alpha_j} = \text{vec}(M)^T A Z_{2d}(x)$$

where $A \in \mathbb{R}^{p^2 \times q}$ is defined as

$$A_{i,j} = \begin{cases} 
1 & \text{if } \alpha_{\text{mod}(i,p)} + \alpha_{[i]p+1} = \beta_j \\
0 & \text{otherwise} 
\end{cases}$$

This then implies that

$$Z_d(x)^T M Z_d(x) = \text{vec}(M)^T A Z_{2d}(x)$$

Hence if we constrain $c = \text{vec}(M)^T A$, this is equivalent to $p(x) = q(x)$
Summarizing, e.g., for Lyapunov stability, we have variables $M > 0, Q > 0$ with the constraint

$$-\text{vec}(M)^T A = \text{vec}(Q)^T AB$$

Feasibility implies stability since

$$V(x) = Z(x)^T Q Z(x) \geq 0$$

$$\dot{V}(x) = \text{vec}(Q)^T A \nabla Z_{2d}(x)$$

$$= \text{vec}(Q)^T AB Z_{2d}(x)$$

$$= -\text{vec}(M)^T A Z_{2d}(x)$$

$$= -Z(x)^T M Z(x) \geq 0$$
YALMIP has SOS functionality

**Link:** YALMIP SOS Manual

To test whether

\[ 4x^4 + 4x^3 y - 7x^2 y^2 - 2xy^3 + 10y^4 \]

is a positive polynomial, we use:

```plaintext
> sdpvar x y
> p = 4*x^4 + 4*x^3*y - 7*x^2*y^2 - 2*x*y^3 + 10*y^4;
> F=[];
> F=[F;sos(p)];
> solvesos(F);
```

To retrieve the SOS decomposition, we use

```plaintext
> sdisplay(p)
> ans =
> '1.7960 * x^2 - 3.0699 * y^2 + 0.6468 * x * y'
> ' - 0.6961 * x^2 - 0.7208 * y^2 - 1.4882 * x * y'
> '0.5383 * x^2 + 0.2377 * y^2 - 0.3669 * x * y'
```
In this class, we will use instead SOSTOOLS

**Link:** SOSTOOLS Website

To test whether

\[ 4x^4 + 4x^3y - 7x^2y^2 - 2xy^3 + 10y^4 \]

is a positive polynomial, we use:

```plaintext
> pvar x y
> p = 4 * x^4 + 4 * x^3 * y - 7 * x^2 * y^2 - 2 * x * y^3 + 10 * y^4;
> prog=sosprogram([x y]);
> prog=sosineq(prog,p);
> prog=sossolve(prog);
```
SOS Programming:
Numerical Example

This also works for matrix-valued polynomials.

\[ M(y, z) = \begin{bmatrix} (y^2 + 1)z^2 & yz \\ yz & y^4 + y^2 - 2y + 1 \end{bmatrix} \]

\[
\begin{bmatrix} (y^2 + 1)z^2 & yz \\ yz & y^4 + y^2 - 2y + 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}
\]

\[
\begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ yz & 0 \\ 0 & 1 \\ 0 & y \\ 0 & y^2 \end{bmatrix}
\]

\[ yz \begin{bmatrix} 1 - y \\ y^2 \end{bmatrix}^T \begin{bmatrix} yz & 1 - y \\ y^2 & y^2 \end{bmatrix} \in \Sigma_s \]
This also works for matrix-valued polynomials.

\[ M(y, z) = \begin{bmatrix} (y^2 + 1)z^2 & yz \\ yz & y^4 + y^2 - 2y + 1 \end{bmatrix} \]

**SOSTOOLS Code:** Matrix Positivity

```matlab
> pvar x y
> M = [(y^2 + 1) * z^2; y * z; y^4 + y^2 - 2 * y + 1];
> prog=sosprogram([y z]);
> prog=sosmatrixineq(prog,M);
> prog=sossolve(prog);
```
An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).

\[
\begin{align*}
\dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\
\dot{y} &= 3x - y
\end{align*}
\]

**SOSTOOLS Code:** Global Stability

```plaintext
> pvar x y
> f = [-y - 1.5*x^2 - 0.5*x^3; 3*x - y];
> prog=sosprogram([x y]);
> Z=monomials([x,y],0:2);
> [prog,V]=sossosvar(prog,Z);
> V = V + 0.0001*(x^4 + y^4);
> prog=soseq(prog,subs(V,[x; y],[0; 0]));
> nablaV=[diff(V,x);diff(V,y)];
> prog=sosineq(prog,-nablaV'*f);
> prog=sossolve(prog);
> Vn=sosgetsol(prog,V)
```

Finds a Lyapunov Function of degree 4.
An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).

\[
\dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3
\]
\[
\dot{y} = 3x - y
\]

**YALMIP Code:** Global Stability

```matlab
> sdpvar x y
> f = [-y - 1.5 * x^2 - .5 * x^3; 3 * x - y];
> [V,Vc]=polynomial([x y],4);
> F=[Vc(1)==0];
> F = [F;sos(V - .00001 * (x^2 + y^2))];
> nablaV=jacobian(V,[x y]);
> F=[F;sos(-nablaV*f)];
> solvesos(F,[],[],[Vc])
```

Finds a Lyapunov Function of degree 4.

- Going forward, we will use mostly SOSTOOLS
There is a third relatively new Parser called SOSOPT

**Link:** [SOSOPT Website](#)

And I can plug my own mini-toolbox version of SOSTOOLS:

**Link:** [DelayTOOLS Website](#)

- However, I don’t expect you to need this toolbox for this class.
An Example of Global Stability Analysis

A controlled model of a jet engine (Derived from Moore-Greitzer).

\[ \dot{x} = -y - \frac{3}{2} x^2 - \frac{1}{2} x^3 \]
\[ \dot{y} = 3x - y \]

This is feasible with

\[
V(x) = 4.5819x^2 - 1.5786xy + 1.7834y^2 - 0.12739x^3 + 2.5189x^2y - 0.34069xy^2 \\
+ 0.61188y^3 + 0.47537x^4 - 0.052424x^3y + 0.44289x^2y^2 + 0.090723y^4
\]
Summary of the SOS Conditions

Proposition 1.

Suppose: \( p(x) = Z_d(x)^T Q Z_d(x) \) for some \( Q > 0 \). Then \( p(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

Refinement 1: Suppose \( Z_d(x)^T P Z_d(x) p(x) = Z_d(x)^T Q Z_d(x) \) for some \( Q, P > 0 \). Then \( p(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

Refinement 2: Suppose \( (\sum_i x_i^2)^q p(x) = Z_d(x)^T Q Z_d(x) \) for some \( P > 0 \), \( q \in \mathbb{N} \). Then \( p(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

Ignore these Refinements

- SOS by itself is sufficient. The refinements are Necessary and Sufficient.
- Almost never necessary in practice...
Problems with SOS

Unfortunately, a Sum-of-Squares representation is not necessary for positivity.

• Artin included ratios of squares.

**Counterexample:** The Motzkin Polynomial

\[ M(x, y) = x^2 y^4 + x^4 y^2 + 1 - 3x^2 y^2 \]

However, \((x^2 + y^2 + 1)M(x, y)\) is a Sum-of-Squares.

\[
(x^2 + y^2 + 1)M(x, y) = (x^2 y - y)^2 + (xy^3 - x)^2 + (x^2 y^2 - 1)^2 \\
+ \frac{1}{4}(xy^3 - x^3 y)^2 + \frac{3}{4}(xy^3 + x^3 y - 2xy)^2
\]